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Equilibrium and Stability of Magnetically Confined Plasmas in the Framework of Extended Magnetohydrodynamic Models, via Hamiltonian Variational Principles

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Ph.D. THESIS

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Περίληψη

Στην παρούσα διατριβή παράγονται εξισώσεις ισορροπίας και ικανές συνθήκες ευστάθειας στάσιμων καταστάσεων μαγνητικά περιορισμένου πλάσματος, μέσω Χαμιλτονιανών μεθόδων. Αυτές πηγάζουν από τη Χαμιλτονιανή δομή της γενικευμένης Μαγνητοϋδροδυναμικής (ΓΜΥΔ), ενός απλοποιημένου, οιονεί ουδέτερου μοντέλου δύο ρευστών που περιλαμβάνει συνεισφορές λόγω ιοντικών ολισθήσεων Hall και ηλεκτρονίων πεπερασμένης αδράνειας. Πιο συγκεκριμένα, ο μη κανονικός Χαμιλτονιανός φορμαλισμός της ΓΜΥΔ προσαρμόζεται για την περιγραφή συστημάτων με συνεχή χωρική συμμετρία καθώς η τρισδιάστατη αγκύλη Poisson ανάγεται στην αντίστοιχη ελικοειδώς συμμετρική αγκύλη. Η ελικοειδής συμμετρία αποτελεί μία γενικευμένη περίπτωση η οποία περιέχει τόσο τη μεταφορική όσο και την αξονική συμμετρία ως υποπεριπτώσεις. Η τελευταία παρουσιάζει ιδιαίτερο ενδιαφέρον για τη μελέτη του πλάσματος σε τοροειδή συστήματα μαγνητικού περιορισμού, όπως το Tokamak, αλλά και του αστροφυσικού πλάσματος. Μέσω της ελικοειδώς συμμετρικής αγκύλης υπολογίζουμε τα αντίστοιχα συναρτησιακά Casimir, τα οποία μετατίθενται με κάθε αυθαίρετο συναρτησιακό των δυναμικών μεταβλητών και ως εκ τούτου αποτελούν αναλλοίωτες ποσότητες. Τα Casimirs και το συναρτησιακό της Χαμιλτονιανής χρησιμοποιούνται για την εφαρμογή της παραλλακτικής αρχής energy-Casimir απο την οποία προκύπτουν γενικευμένες εξισώσεις ισορροπίας που στη συνέχεια γράφονται στη μορφή ενός συστήματος τύπου Grad-Shafranov-Bernoulli. Επίσης μελετώνται ειδικές περιπτώσεις, όπως το αντίστοιχο αξονικά συμμετρικό σύστημα, για το οποίο παράγεται η συνθήκη ελλειπτικότητας. Επιπρόσθετα, αμελώντας την αδράνεια των ηλεκτρονίων υπολογίζουμε μια αριθμητική, αξονικά συμμετρική ισορροπία σε συνάρτηση με τους λεγόμενους βελτιωμένους τρόπους περιορισού που παρατηρούνται στο Tokamak. Όσον αφορά την ευστάθεια, στα πλαίσια της μη κανονικής Χαμιλτονιανής περιγραφής, εξάγουμε ικανές συνθήκες ευστάθειας χρησιμοποιώντας τόσο τη μέθοδο energy-Casimir όσο και τη μέθοδο των δυναμικά προσβάσιμων διαταραχών. Η πρώτη εφαρμόζεται για τη μελέτη της ευστάθειας ισορροπιών Tokamak με τοροειδή ροή, στο όριο της μαγνητοϋδροδυναμικής Hall. Επιπλέον, εφαρμόζοντας τη Λαγκρανζιανή περιγραφή για τη δυναμική των ρευστών παράγονται ικανά κριτήρια ευστάθειας, κάτω από Λαγκρανζιανές μετατοπίσεις, για το γενικό, οιονεί ουδέτερο μοντέλο δύο ρευστών και για τη μαγνητοϋδροδυναμική Hall. Τα χαρακτηριστικά της κάθε μεθόδου συζητούνται εμβριθώς. Τέλος, προτείνουμε μια εναλλακτική περιγραφή της ασυμπίεστης ΓΜΥΔ μέσω τριγραμμικών αγκυλών και μια ευρετική μέθοδο για την κατασκευή διδιάστατων δυναμικών μοντέλων επιβάλλοντας εκ των προτέρων τη διατήρηση της Χαμιλτονιανής και των αναλλοίωτων Casimir.

Abstract

In this thesis, equilibrium equations and sufficient stability criteria for stationary states of magnetically confined plasmas are derived via Hamiltonian methods. These methods originate from the Hamiltonian structure of extended Magnetohydrodynamics (XMHD) a simplified, quasineutral, two-fluid model that takes into account contributions due to ion Hall drifts and finite electron inertia. More specifically, the noncanonical Hamiltonian formulation of XMHD is adapted for the description of systems with continuous spatial symmetry upon reducing the three-dimensional Poisson bracket to the corresponding helically symmetric bracket. Helical symmetry is a generic case including both translation and axial symmetry as special cases. The latter is particularly interesting for the study of toroidal systems for magnetic confinement, such as the Tokamak, and also for astrophysical plasmas. By the helically symmetric Poisson bracket, we compute the corresponding Casimir functionals, that Poisson-commute with any arbitrary functional of the dynamical variables, thus being invariant quantities. The Casimirs, along with the Hamiltonian, are then used to employ the energy-Casimir variational principle resulting in a set of equilibrium equations that are cast in the form of a Grad-Shafranov-Bernoulli (GSB) system. Special cases are considered, e.g., the corresponding axisymmetric system of equations, whose ellipticity condition is derived. Moreover, neglecting electron inertia, we compute a numerical, axisymmetric equilibrium in connection with the so-called improved confinement modes observed in Tokamaks. Regarding stability, within the noncanonical Hamiltonian framework, we obtain sufficient stability criteria using the energy-Casimir and the dynamically accessible stability methods. The former is exploited for assessing the stability of Tokamak, Hall MHD (HMHD) equilibria with toroidal rotation. In addition, employing the Lagrangian description for fluid dynamics, sufficient stability criteria under Lagrangian displacements are derived for the generic, quasineutral, two-fluid model and also for HMHD. The characteristics of each method are thoroughly discussed. Lastly, we propose an alternative description of incompressible XMHD in terms of trilinear brackets and a heuristic procedure for constructing two-dimensional dynamical models upon imposing a priori the conservation of the Hamiltonian and Casimir invariants.

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Chapter 1

Introduction

1.1 Plasma Physics, thermonuclear fusion and magnetic confinement

As it is well known from classical electrodynamics, the electromagnetic field is described by the electromagnetic tensor, which is computed upon solving Maxwell's equations. A prerequisite for this computation though, is the knowledge of the four-current, i.e. the electric charge and current densities. Therefore, we need to know how the various particle motions generate this four-current. Here the problem of self-consistency arises, that is the motions of the charged particles (sources) must be followed in the fields they generate and also in those that are imposed externally. Hence, it becomes quite clear that one needs a self-consistent theory of interaction between matter and electromagnetic fields. In nature it is rather unusual to find systems of charged particles simple enough so as this theory to be provided by electrodynamics and ordinary dynamics alone, since we usually have to deal with collections of many particles and tracking the motion of every single particle is practically impossible. In extraterrestrial environments and laboratory experiments, these sources are often found in media being in plasma state, i.e. media consisting of collections of many charged particles of different species, exhibiting collective behavior while being influenced mainly by the electromagnetic forces. Plasma Physics provides a framework for describing the interaction between such sources and electromagnetic fields, employing several theories and methodologies, which differentiate with respect to their assumptions and the level of refinement aiming to provide closures to Maxwell's equations.

Beyond this fundamental role of plasma physics, which renders it the natural framework for astrophysical studies since most of the baryonic matter in the Universe is in plasma state, there are a lot of practical reasons to study the bizarre behavior of the plasmas. The most important reason is thermonuclear fusion, since under thermonuclear conditions the matter is in plasma state. After the development of the hydrogen bomb, the pursuit for a controlled release and exploitation of fusion energy was intensified. Peaceful applications of fusion energy can potentially provide the necessary energy for mankind's prosperity, surmounting crucial environmental, economical and geopolitical problems related to the increasing energy needs and consumption. Despite the remarkable progress that has been made from the 1950's to nowadays, fusion energy remains still an unaccomplished goal. However, its important comparative advantages with respect to other energy sources still push the research forward with a lot of experimental fusion devices running all over the world. The main advantages of fusion is the large fuel reserves (Deuterium can be found in sea water) and the reduced environmental impact, because of the very limited amount of radioactive disposals with comparatively short-lived radioactivity and the elimination of the possibility of severe nuclear accidents since no fissile materials are used. Any accident in a fusion reactor will provoke just the reactor shutdown. Over the last seven decades of fusion research however, it has been understood that achieving controlled thermonuclear fusion is a rather difficult task. The reason is the bizarre and many times unpredictable behavior that plasma exhibits, especially when we try to keep it confined.

To achieve self-sustained fusion, plasma confinement is crucial. To have a Deuterium - Tritium (D-T) nuclear fusion process

$$ {}^{2}_{1}D + {}^{3}_{1}T \rightarrow {}^{4}_{2}He\,(3.5\,MeV) + n\,(14.1\,MeV)\,, $$

we need to heat the fuel up to $150\times10^{6}\ K$, a temperature that exceeds the temperature at the center of the sun. Moreover, the plasma has to be confined for sufficient time. However, such a hot material is difficult to be confined effectively, i.e. to reduce the loss of mass and thermal energy so as the Lawson criterion [1] $n\tau_e \geq 1.5\times10^{20}s/m^3$, where n is the particle density and τ_e is the confinement time, to be satisfied. The Lawson criterion essentially states that the rate of energy production must exceed the rate of energy loss. To establish effective confinement, appropriate configurations of strong magnetic fields are utilized to hold and compress the plasma. The difficulty is that confinement should be sustained for sufficient amount of time at sufficiently high temperatures. This challenge is the main force behind the flourishing of Plasma Physics and the development of a variety of models for describing the plasmas.

Research in magnetic confinement is nowadays focused in two main directions. The first and most promising is the Tokamak concept. The Tokamak is a toroidal configuration which enrolls large coils to produce a toroidal magnetic field and a solenoid in order to induce a toroidal current thus establishing a poloidal magnetic field. The poloidal field produces a rotational transform of magnetic field lines, which is necessary for canceling grad-B and curvature drifts of particles to the wall. The second important device is the Stellarator, which is also toroidal but has continuous operation since the rotational transform is produced via appropriately bending the external coils. Despite its pulse operation the Tokamak remains the best candidate for a fusion reactor due to its simplicity and the observed improved confinement modes characterized by high plasma beta, i.e. high ratio of plasma pressure over magnetic pressure

$\beta = 2\mu_0 P/(B^2)$, that indicates good quality of confinement.

These high confinement modes, achieved via the so-called L-H (Low-High) transitions occur when the plasma heating power exceeds a critical value [2]. It has been observed that those transitions are accompanied with an increase of the sheared $\mathbf{E}\times\mathbf{B}$ flow and the formation of transport barriers. It is believed that there is a relation between improved confinement and sheared flows because the latter effectively reduces radial turbulent transport, which is the main mechanism behind heat conduction and therefore the loss of energy. For many years in fusion research, flows where disregarded. For example, the main tool for almost every study on Tokamak equilibria, the Grad-Shafranov equation (e.g. see [3]), was employed in its simplest form, i.e. neglecting macroscopic flows. Over the last decades though, due to the important role of macroscopic mass flows, which can be externally driven or self-generated, there is an increasing interest for the study of flows in fusion experiments, from constructing stationary states to the study of stability, turbulence and dynamics in general. The aims of the present study are oriented in this direction, incorporating not only flow effects but additionally effects that emerge due to the existence of different species of particles. To value the importance of such a description we need to understand the framework within which fusion research takes place. For this reason in the section below we present some plasma models, from kinetic theory to single fluid Magnetohydrodynamics [4], which are important not only for fusion research but also for Astrophysics.

1.2 Plasma modeling

1.2.1 From kinetic to multi-fluid description

The plasmas consist of a very large number of charged particles which move upon interacting with electromagnetic fields, imposed externally and generated by the plasma itself. It is clear therefore that keeping track of every single particle motion is practically impossible. Also, the initial conditions cannot be known with sufficient accuracy. For these reasons we often adopt a statistical description in terms of distribution functions for each particle species, which change while the plasma interacts with the EM fields. The evolution of the distribution function for each particle species is described by Boltzmann's equation

$$\partial_t f_s + \boldsymbol{v}\cdot\nabla_{\mathbf{x}} f_s + \frac{e_s}{m_s}(\mathbf{E}+\boldsymbol{v}\times\mathbf{B})\cdot\nabla_{\boldsymbol{v}} f_s = (\partial_t f_s)_c\ , \tag{1.1}$$

where e_s and m_s and are the species charge and mass, respectively; $\mathbf{B}$ and $\mathbf{E}$ are the magnetic and electric fields, respectively and $(\partial_t f_s)_c$ represents the change of the distribution functions due to binary Coulomb collisions. Note that $\boldsymbol{v}$ refers to particle velocity. For an ideal, collisionless plasma the rhs of (1.1) can be neglected resulting in the so-called Vlasov equation. We are interested in the ideal case not only because it is a simplification which allows for the nice conservation and Hamiltonian properties to emerge, but also because this case is physically justified by Spitzer's law of conductivity which states that the collision frequency between particles scales as $T^{-3/2}$. Hence, for the extremely hot Tokamak plasmas for example, the importance of collisional effects is significantly degraded at least in the framework of some particular applications.

Even the statistical description though can become very complicated since the evolution of the distribution functions takes place in a six-dimensional phase space. Also there are phenomena for which a kinetic description provides excessive information which is not really needed for an adequate description. Hence, it is very common to resort to simpler models such as the multi-fluid model and the ordinary Magnetohydrodynamics. The ideal, multi-fluid equations are derived upon taking the velocity moments of Vlasov's equation, in view of the following definitions

$$n_s(\mathbf{x},t) = \int d^3v f_s(\mathbf{x},\boldsymbol{v},t)\,, \tag{1.2}$$

$$\mathbf{v}_s(\mathbf{x},t) = n_s^{-1}(\mathbf{x},t)\int d^3v\, \boldsymbol{v} f_s(\mathbf{x},\boldsymbol{v},t)\,, \tag{1.3}$$

$$\mathbf{P}_s = m_s\int d^3v\,(\mathbf{v}-\mathbf{v}_s)(\boldsymbol{v}-\mathbf{v}_s)f_s(\mathbf{x},\boldsymbol{v},t) = p_s\mathbb{I}+\boldsymbol{\pi}_s\,, \tag{1.4}$$

$$\mathbf{Q}_s = \frac{m_s}{2}\int d^3v|\boldsymbol{v}-\mathbf{v}_s|^2(\boldsymbol{v}-\mathbf{v}_s)f_s(\mathbf{x},\boldsymbol{v},t)\,, \tag{1.5}$$

where n_s are the species particle densities and $\mathbf{P}_s$ represent the species pressure tensors. Here, $p_s = (1/3)Tr(\mathbf{P}_s)$ and $\boldsymbol{\pi}_s$ are the scalar pressures and generalized viscosity tensors, respectively and $\mathbf{Q}_s$ the species energy flux density. Taking the zeroth order velocity moment of Vlasov's equation we can easily find the continuity equation

$$\partial_t n_s + \nabla\cdot(n_s\mathbf{v}_s) = 0\,. \tag{1.6}$$

Similarly, if we consider the first order velocity moment of the collisionless counterpart of (1.1), multiplying with velocity and integrating in velocity space, then upon exploiting (1.6) we end up with the following momentum equations

$$m_s n_s\left(\partial_t\mathbf{v}_s + \mathbf{v}_s\cdot\nabla\mathbf{v}_s\right) = e_s n_s\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right) - \nabla\cdot\mathbf{P}_s\,, \quad s=i,e\,. \tag{1.7}$$

Finally, the contracted second moment gives the energy conservation equation

$$\frac{3}{2}\frac{dp_s}{dt} + \frac{5}{2}p_s\nabla\cdot\mathbf{v}_s + \boldsymbol{\pi}_s:\nabla\mathbf{v}_s + \nabla\cdot\mathbf{Q}_s = 0\,. \tag{1.8}$$

If the distribution functions f_s are isotropic, e.g. Maxwellians, then the non-diagonal elements in the pressure tensors $\mathbf{P}_s$ vanish and the diagonal terms are equal, therefore the tensors can be replaced by the scalar pressures p_s, $s = i, e$. In addition, if we assume that there is no heat transfer between the fluid elements then Eq. (1.8) becomes

$$\frac{dp_s}{dt} + \frac{5}{3} p_s \nabla \cdot \mathbf{v}_s = 0\,. \tag{1.9}$$

If the internal energies of the fluids depend only on the particle density (barotropic fluids) then (1.9) is satisfied for $p_s = c n_s^{5/3}$, whereas if they depend also on the entropy (baroclinic fluids) then (1.9) yields

$$p_s = A_s(s_s) n_s^{5/3}\,, \tag{1.10}$$

$$\frac{ds_s}{dt} = 0\,, \tag{1.11}$$

where s_s are the species specific entropies. In this thesis we consider barotropic fluids, i.e. $s_s = const$. Note that in general, $p_s = A(s_s) n_s^{\Gamma}$, where Γ is the adiabatic index.

Before proceeding to the derivation of our fluid model we need to make some remarks regarding the fluid description of the plasmas. The fluid equations are suitable for describing the macroscopic behavior of the plasmas, i.e. the microscopic motions are eliminated by averaging the physical quantities. This is a sufficient approximation when collisions are prominent, providing a physical mechanism for the averaging of the physical quantities. As mentioned earlier though, in a hot plasma, collisions are rather rare and therefore this “fluidifying” mechanism is provided by the strong magnetic fields, which affect the motion of the charged particles perpendicular to the magnetic field lines. This is indeed the case in a plethora of applications concerning fusion or astrophysical plasmas where strong magnetic fields are present. However, when we are dealing with rarefied or high temperature plasmas embedded in weak magnetic fields or when considering the parallel dynamics, then a kinetic description should be employed.

1.2.2 Extended Magnetohydrodynamics

The two-fluid equations can be written in the simplified form of single-fluid equations upon assuming quasineutrality and changing variables from the two fluid velocities to a center of mass velocity and the current density. Ordinary Magnetohydrodynamics (MHD) [4] is the most widely employed fluid model, which neglects the existence of multi-fluid components. Despite the boldness of its fundamental assumptions, MHD is a very successful theory for describing astrophysical and laboratory plasmas, since in many cases it provides an adequate framework. This is indeed the case if

- the length scale of the system is much larger than the Debye length (quasineutrality) and the ion and electron gyroradii and skin depths i.e. $L \gg \lambda_D, \rho_s, \lambda_s$, $s = i, e$.

- the characteristic time scale is much longer than the inverses of the plasma and cyclotron frequencies: $\tau \gg \omega_{ps}^{-1}, \Omega_s^{-1}$, $s = i, e$.

- the Alfvén speed is small compared to the speed of light $v_A \ll c$, therefore the displacement current in the Ampere's law can be neglected.

The assumption $L \gg \lambda_D$ is necessary for quasineutrality, while $L \gg \rho_s, \lambda_s$ and the second assumption are necessary so as two-fluid effects can be neglected. It is not immediately obvious but the third assumption is related also to quasineutrality, which turns out to be a necessary condition for a nonrelativistic fluid theory to be compatible with Maxwell's equations (see [5]). This is because nonrelativistic fluid equations are Galilean invariant, whereas Maxwell's equations are Lorentz invariant. To be consistent with a nonrelativistic fluid description, Maxwell's equations must become Galilean invariant as well. This is done upon considering a nonrelativistic limit by assuming that the characteristic speed is very small compared to the speed of light, which has as a consequence the displacement current to be negligible. However, this assumption also implies quasineutrality. To see this let us write the Gauss law for the electric field in Alfvén units

$$\nabla \cdot \mathbf{E} = \frac{L}{\lambda_i} \frac{c^2}{v_A^2} (n_i - n_e) \,, \tag{1.12}$$

where λ_i is the ion skin depth. In the limit $v_A/c \ll 1$ the rhs diverges unless the particle density difference is at least $n_i - n_e \sim \mathcal{O}(v_A^2/c^2)$ that is the very definition of quasineutrality.

The MHD model is described by the following equations

$$\partial_t \rho = -\nabla \cdot (\rho \mathbf{v}) \,, \tag{1.13}$$

$$\rho(\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v}) = -\nabla p + \mathbf{J} \times \mathbf{B} \,, \tag{1.14}$$

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{v} \times \mathbf{B}) \,, \tag{1.15}$$

$$\text{an equation of state e.g. adiabatic} \tag{1.16}$$

However, if the time scales or/and length scales become comparable to the characteristic time or/and lengths of the charged particle motions then the single fluid MHD becomes clearly inadequate. This could indeed be the case when there exist fast oscillating modes propagating into the plasma or small length scale structures, such as current sheets. Even in the absence of such characteristics, small length scale structures are indeed present since in general, due to the particle drifts, the ions and even the much lighter electrons get separated from the magnetic field lines. This separation distance is important for transport phenomena and consequently it would be preferable to work within a context which provides estimates of this length. Such a context is the two-fluid theory. The first step towards a generalization of ordinary MHD so as to include two fluid effects is the inclusion of the Hall term in the MHD Ohm's law. This leads to the so-called Hall MHD (HMHD), which assumes however massless electrons. Incorporating electron inertia into the HMHD model leads to extended MHD (XMHD) [6]. It must be emphasized here that the key assumption for the derivation of both HMHD and XMHD is quasineutrality. Indeed, if the first of the three conditions described above is satisfied, that is the length scales are much larger than the Debye length, which is characteristic of the screening of electric fields into the plasma, we are allowed to assume quasineutrality, $n_i \approx n_e \approx n$. In general, for a single ion - electron plasma, we can define a center of mass velocity as follows

$$\mathbf{v} := \frac{m_i n_i \mathbf{v}_i + m_e n_e \mathbf{v}_e}{m_i n_i + m_e n_e}\,, \tag{1.17}$$

which upon assuming $n_i = n_e = n$ reduces to

$$\mathbf{v} := \frac{m_i \mathbf{v}_i + m_e \mathbf{v}_e}{m_i + m_e}\,. \tag{1.18}$$

With this definition one can start from the two-fluid system to derive the simplified versions of HMHD and XMHD. Multiplying the continuity equations with the corresponding masses, then adding the results and using Eq. (1.18), one deduces a single fluid continuity equation of the form

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0\,, \tag{1.19}$$

where $\rho := (m_i + m_e)n$. In addition, upon identifying that

$$\mathbf{v}_e = \mathbf{v} - \frac{m_i}{m}\frac{\mathbf{J}}{en}\,, \tag{1.20}$$

$$\mathbf{v}_i = \mathbf{v} + \frac{m_e}{m}\frac{\mathbf{J}}{en}\,, \tag{1.21}$$

which can easily be corroborated combining Eq. (1.18) with $\mathbf{J} = en(\mathbf{v}_i - \mathbf{v}_e)$ and adding the two momentum equations, we deduce a single fluid momentum equation which reads as follows

$$\begin{aligned}\partial_t \mathbf{v} = \mathbf{v} \times \nabla \times \mathbf{v} + \frac{m_i m_e}{m^2}\frac{\mathbf{J}}{en} \times \nabla \times \frac{\mathbf{J}}{en} \\ -\nabla\left[h + \frac{|\mathbf{v}|^2}{2} + \frac{m_i m_e}{2m^2}\left|\frac{\mathbf{J}}{en}\right|^2\right] + (mn)^{-1}\mathbf{J} \times \mathbf{B}\,.\end{aligned} \tag{1.22}$$

If we multiply the ion momentum equation with m_e and the corresponding electron equation by m_i and then subtract the resulting equations, we find a generalized Ohm's law. Essentially it is a momentum equation for the difference of the ion and electron momenta. However, it plays the role of an Ohm's law since it relates the electric field $\mathbf{E}$ with the current density $\mathbf{J}$. One can show that this equation is of the form

$$\begin{aligned} \mathbf{E} + \mathbf{v} \times \mathbf{B}^* = \frac{m_i m_e}{em} \Bigg\{ \partial_t \left(\frac{\mathbf{J}}{en} \right) \\ - \frac{\mathbf{J}}{en} \times \nabla \times \mathbf{v} + \nabla \left(\frac{\mathbf{v} \cdot \mathbf{J}}{en} + \frac{m_e^2 - m_i^2}{2m^2} \left| \frac{\mathbf{J}}{en} \right|^2 \right) \Bigg\} \\ - \frac{m_e^2 - m_i^2}{m^2} \left(\frac{\mathbf{J}}{en} \times \mathbf{B}^* \right) + \frac{1}{men} \nabla (m_e p_i - m_i p_e) \,, \end{aligned} \tag{1.23}$$

where $\mathbf{B}^*$ is a generalized “magnetic field", modified by electron inertia

$$\mathbf{B}^* = \mathbf{B} + \frac{m_i m_e}{me} \left(\nabla \times \frac{\mathbf{J}}{en} \right) . \tag{1.24}$$

From Faraday's law of induction, $\partial_t \mathbf{B} = -\nabla \times \mathbf{E}$, we can find a dynamical equation governing the evolution of this field

$$\begin{aligned} \partial_t \mathbf{B}^* = \nabla \times \left[\mathbf{v} \times \mathbf{B}^* + \frac{m_i m_e}{em} \frac{\mathbf{J}}{en} \times (\nabla \times \mathbf{v}) + \frac{m_e - m_i}{m} \frac{\mathbf{J}}{en} \times \mathbf{B}^* \right] \\ + \frac{1}{emn^2} \nabla n \times (m_e \nabla p_i - m_i \nabla p_e) \,. \end{aligned} \tag{1.25}$$

We showed that the quasineutral two-fluid model can be written as a dynamical system with characteristics similar to MHD, that is a single continuity equation (1.19), a momentum equation for a center-of-mass velocity (1.22) and an induction equation governing the evolution of a magnetic-like field (1.25). In most of the studies employing extended MHD, a reduced version of the quasineutral model derived above is utilized. This reduction involves an expansion in the smallness of the electron over the ion mass ratio $\mu := m_e/m_i$ and keeping terms of $\mathcal{O}(\mu^0)$. For this expansion to be performed, one needs to write the dynamical equations in dimensionless form. This is effected through the so-called Alfvén normalization

$$\begin{aligned} &\bar{n} = n/n_0 \,, \quad \bar{t} = t/\tau_A \,, \quad \bar{\mathbf{B}} = \mathbf{B}/B_0 \,, \\ &\bar{\mathbf{J}} = \mathbf{J}/(B_0/\ell\mu_0) \,, \quad \bar{\nabla} = \ell\nabla \,, \quad \bar{\mathbf{A}} = \mathbf{A}/(\ell B_0) \,, \\ &\bar{\mathbf{E}} = \mathbf{E}/(v_A B_0) \,, \quad \bar{\Phi} = \Phi/(\ell v_A B_0) \,, \quad \bar{p_s} = p_s/(B_0^2/\mu_0) \,, \end{aligned} \tag{1.26}$$

where $\mathbf{A}$ and Φ are the electromagnetic vector and scalar potentials; ℓ, n_0 and B_0 are reference length, particle density and magnetic field, respectively; $v_A = B_0/\sqrt{\mu_0 m_i n_0}$

is the Alfvén speed and $\tau_A = \ell/v_A$ is the Alfvén time. Henceforth, to simplify notation, the bars will be omitted on the understanding that all appearing quantities are normalized as described above. Upon normalizing Eqs. (1.22)–(1.23) and neglecting $\mathcal{O}(\mu)$ terms we find

$$\partial_t \mathbf{v} = -\nabla\left(h + \frac{|\mathbf{v}|^2}{2} - \frac{d_e^2}{2}\frac{|\mathbf{J}|^2}{\rho^2}\right) + \mathbf{v}\times\nabla\times\mathbf{v} + \mathbf{J}\times\mathbf{B}^* \,, \tag{1.27}$$

$$\mathbf{E} + \mathbf{v}\times\mathbf{B} = -d_i\frac{\nabla p_e}{\rho} + d_i\frac{\mathbf{J}\times\mathbf{B}^*}{\rho}$$
$$+d_e^2\left[\partial_t\left(\frac{\mathbf{J}}{\rho}\right) - \rho^{-1}\mathbf{J}\times\nabla\times\mathbf{v} + \nabla\left(\frac{\mathbf{v}\cdot\mathbf{J}}{\rho} - \frac{d_i}{2}\frac{|\mathbf{J}|^2}{\rho^2}\right)\right] . \tag{1.28}$$

Considering a barotropic plasma i.e. the specific enthalpy h and the pressures p, p_e are functions of the particle density only, the induction equation becomes

$$\partial_t \mathbf{B}^* = \nabla\times\left(\mathbf{v}\times\mathbf{B}^* - d_i\frac{\mathbf{J}\times\mathbf{B}^*}{\rho} + d_e^2\frac{\mathbf{J}\times\nabla\times\mathbf{v}}{\rho}\right) , \tag{1.29}$$

where

$$\mathbf{B}^* := \mathbf{B} + d_e^2\nabla\times\frac{\mathbf{J}}{\rho} \,, \tag{1.30}$$

and $d_i = c/(\omega_{pi}L)$, $d_e = c/(\omega_{pe}L)$ are the normalized ion and electron skin depths, respectively.

1.3 Hamiltonian description of ideal fluid models

1.3.1 Canonical and noncanonical Hamiltonian mechanics

In the framework of canonical Hamiltonian mechanics, conservative dynamical systems with N degrees of freedom are described by a set of generalized coordinates q^i, $i = 1, ..., N$ and a set of generalized momenta π_i, $i = 1, ...N$, which are associated with the time derivatives of q^i's via

$$\pi_i = \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \,, \tag{1.31}$$

where $\mathcal{L} = \mathcal{L}[q, \dot{q}, t]$ is the Lagrangian of the system. The dynamics is then described by the Hamilton's equations which read as

$$\dot{q}^i = \frac{\partial \mathcal{H}}{\partial \pi_i}$$
$$\dot{\pi}_i = -\frac{\partial \mathcal{H}}{\partial \dot{q}^i} \,, \tag{1.32}$$

where $\mathcal{H}[q,\pi,t] := \pi_i\dot{q}^i - \mathcal{L}[q,\dot{q},t]$ is the Hamiltonian of the system. Defining the phase-space coordinates

$$z^i = \begin{cases} q^i, & i = 1,...,N \\ \pi_{i-N}, & i = N+1,...,2N \end{cases} . \tag{1.33}$$

Hamilton's equations (1.32) assume the following compact, covariant form

$$\dot{z}^i = \mathcal{J}_c^{ij}\frac{\partial \mathcal{H}}{\partial z^j}, \tag{1.34}$$

where $\mathcal{J}_c$ is the Poisson operator given by the co-symplectic form

$$\mathcal{J}_c = \begin{pmatrix} 0_N & I_N \\ -I_N & 0_N \end{pmatrix}, \tag{1.35}$$

with I_N being the $N \times N$ identity matrix. The canonical Poisson bracket between two arbitrary functionals $f[z]$, $g[z]$ is defined as follows

$$[f,g] = \frac{\partial f}{\partial z^i}\mathcal{J}_c^{ij}\frac{\partial g}{\partial z^j}. \tag{1.36}$$

Poisson brackets satisfy certain rules, they are bilinear, antisymmetric, they obey the Leibniz rule and furthermore satisfy the Jacobi identity, which reads as follows

$$[f,[g,h]] + [h,[f,g]] + [g,[h,f]] = 0. \tag{1.37}$$

These ingredients more or less consist the standard canonical Hamiltonian formulation of mechanics, as is described in textbooks, for example [7]. However, when dealing with continua, e.g. fluids (their phase space is infinite dimensional), the natural framework for the description of their dynamics is the so-called Eulerian viewpoint, describing the fluid motion by measuring the change of physical quantities at fixed point. Eulerian variables in continuum models are in general noncanonical in the sense that they do not constitute canonical pairs and the transformation that connects them with the material variables, that are canonical, introduce into the Poisson operator an explicit dependence on the phase space variables. This means that the co-symplectic form (1.35) ceases to be an appropriate Poisson operator in the Eulerian framework. To see how this transpires (see [8]) it is sufficient to stick in the finite dimensional treatment and consider a coordinate transformation $z^i \to u^i = u^i(z)$. Hamilton's equations (1.34) become

$$\dot{u}^m = \frac{\partial u^m}{\partial z^i}\mathcal{J}_c^{ij}\frac{\partial u^n}{\partial z^j}\frac{\partial \tilde{\mathcal{H}}[u]}{\partial u^n} =: \mathcal{J}^{mn}\frac{\partial \tilde{\mathcal{H}}[u]}{\partial u^n}, \tag{1.38}$$

thus it becomes clear that the new Poisson operator $\mathcal{J}$ in general depends on the phase space variables i.e. $\mathcal{J} = \mathcal{J}[u]$. It retains however the property of antisymmetry and satisfies Jacobi identity, which in tensorial form read as follows

$$\begin{aligned} &\mathcal{J}^{ij} = -\mathcal{J}^{ji}\,, \\ &\mathcal{J}^{im}\frac{\partial \mathcal{J}^{jk}}{\partial u^m} + \mathcal{J}^{jm}\frac{\partial \mathcal{J}^{ki}}{\partial u^m} + \mathcal{J}^{km}\frac{\partial \mathcal{J}^{ij}}{\partial u^m} = 0\,. \end{aligned} \tag{1.39}$$

The noncanonical Hamilton's equations are given by

$$\dot{u}^i = \mathcal{J}^{ij}(u)\frac{\partial \tilde{\mathcal{H}}[u]}{\partial u^j}\,, \tag{1.40}$$

and the noncanonical Poisson bracket is

$$[f,g] = \frac{\partial f}{\partial u^i}\mathcal{J}^{ij}(u)\frac{\partial g}{\partial u^j}\,. \tag{1.41}$$

The temporal evolution of a functional $\mathcal{F}$ is then given by

$$\partial_t \mathcal{F} = [\mathcal{F}, \mathcal{H}]\,, \tag{1.42}$$

A straightforward generalization to infinite dimensions is possible

$$\{F,G\} = \int d\mu\, \frac{\delta F}{\delta u^i}\mathcal{J}^{ij}\frac{\delta G}{\delta u^j}\,, \tag{1.43}$$

where μ are spatial or in general Eulerian coordinates and $\delta F/\delta u$ denotes the functional derivative defined via the variation of the functional F

$$\delta F = \lim_{\epsilon \to 0}\frac{F[u+\epsilon\delta u] - F[u]}{\epsilon} = \int d\mu \frac{\delta F}{\delta u}\delta u\,. \tag{1.44}$$

In view of (1.43) the fundamental properties (1.39) take the form

$$\{F,G\} = -\{G,F\}\,, \tag{1.45}$$

$$\{F,\{G,H\}\} + \{H,\{F,G\}\} + \{H,\{F,G\}\} = 0\,. \tag{1.46}$$

1.3.2 Casimir invariants and the energy-Casimir variational principle

A characteristic of noncanonical Poisson operators is that they are degenerate and inhomogeneous, that is, the kernel, $ker(\mathcal{J})$, contains nonzero elements and its dimension may depend on the position in the phase space. The nonzero elements of the Poisson kernel are phase space gradients of topological invariants called Casimirs, i.e.

$$\mathcal{J}^{ij}(\boldsymbol{u})\frac{\partial \mathcal{C}_k}{\partial u^j} = 0\,, \tag{1.47}$$

hence, $\{\mathcal{C},\mathcal{F}\}=0\ \forall\mathcal{F}$; in other words they commute with every arbitrary functional $\mathcal{F}$ defined on the phase space $\mathcal{P}$. From (1.47) one can easily see that the quantities $\mathcal{C}$ are constants of motion since

$$\dot{\mathcal{C}}^k=\{\mathcal{C},\mathcal{H}\}=\frac{\partial\mathcal{C}_k}{\partial u^i}\mathcal{J}^{ij}(\boldsymbol{u})\frac{\partial\mathcal{H}}{\partial u^j}=0\,, \tag{1.48}$$

for every Hamiltonian $\mathcal{H}$. Equation (1.47) implies that there are surfaces onto which the phase space trajectories of the noncanonical Hamiltonian system are confined. These surfaces are the level sets of the Casimir invariants i.e. surfaces defined by $\mathcal{C}=const.$. We understand therefore that the Casimir invariants play an important role in dynamical evolution, since they restrict the phase space trajectories. Their role is also important for the evaluation of equilibrium points for which $\partial_t\mathcal{F}=0\ \forall\mathcal{F}$. From (1.42) one deduces that $\partial_t\mathcal{F}=0$ is equivalent to

$$\mathcal{J}^{ij}\frac{\partial\mathcal{H}}{\partial u^j}=0\,. \tag{1.49}$$

However, due to (1.47) we understand that (1.49) does not give all the possible equilibria since

$$\mathcal{J}^{ij}\frac{\partial}{\partial u^j}\left(\mathcal{H}+\sum_k\mathcal{C}_k\right)=0\,, \tag{1.50}$$

as well. In the infinite dimensional case, where $\{F,G\}=\int d\mu\,\frac{\delta F}{\delta u^i}\mathcal{J}^{ij}\frac{\delta G}{\delta u^j}$, the partial derivatives are replaced by functional derivatives, i.e.

$$\mathcal{J}^{ij}\frac{\delta}{\delta u^j}\left(\mathcal{H}+\sum_k\mathcal{C}_k\right)=0\,, \tag{1.51}$$

It is evident then that phase space points, $\boldsymbol{u}_e$, satisfying

$$\delta\left(\mathcal{H}+\sum_k\mathcal{C}_k\right)[\boldsymbol{u}_e]=0\,, \tag{1.52}$$

are equilibrium points. Note thought that the converse is not true, i.e. not all equilibrium points are solutions to Eq. (1.52) (see [8, 9]). Equation (1.52) is the mathematical expression of the so-called energy-Casimir (EC) variational principle and the equilibrium states derived by this are called energy-Casimir equilibria. This variational principle is the core concept of the present thesis, since in the following chapters equilibrium equations for the XMHD model are derived upon extremizing XMHD EC functionals.

1.3.3 Second variation of the EC functional - EC stability

Employing the EC principle for computing equilibria has an undeniable advantage: the variational principle can be utilized for studying the stability of the EC equilibria as well. This can be done upon computing the second order variation of the EC functional, $\mathcal{H}_C := \mathcal{H} - \sum_i \mathcal{C}_i$, at the corresponding equilibrium and investigating if it is of definite sign. If so, then the equilibrium is linearly stable. To see how this transpires we need to examine the linear dynamics of a noncanonical Hamiltonian system. Since $\mathcal{C}$ is a constant of motion for the general nonlinear dynamics, we can freely write

$$\dot{u}^i = \mathcal{J}^{ij}\frac{\delta \mathcal{H}_C}{\delta u^j}\,, \tag{1.53}$$

without changing the dynamics. The linearized equations of motion, around an equilibrium $\boldsymbol{u}_e$ are obtained upon expanding in Taylor series (up to first order) the equation above, in view of $\boldsymbol{u} = \boldsymbol{u}_e + \epsilon\delta\boldsymbol{u}$, where ϵ is a small parameter. By doing so we find

$$\delta\dot{u}^i = \mathcal{J}^{ij}(\boldsymbol{u}_e)\frac{\delta^2\mathcal{H}_C(\boldsymbol{u}_e)}{\delta u^j\delta u^k}\delta u^k\,, \tag{1.54}$$

where we have used $\dot{u_e}^i = 0$ and $\delta\mathcal{H}_C(\boldsymbol{u}_e) = 0$. Now let us consider the time derivative of $\delta^2\mathcal{H}_C$ at $\boldsymbol{u}_e$

$$\begin{aligned}\frac{d}{dt}\delta^2\mathcal{H}_C &= \frac{d}{dt}\int d\mu \frac{\delta^2\mathcal{H}_C}{\delta u^j\delta u^k}(\boldsymbol{u}_e)\delta u^j\delta u^k \\ &= \int d\mu \frac{\delta^2\mathcal{H}_C}{\delta u^j\delta u^k}(\boldsymbol{u}_e)(\delta\dot{u}^j\delta u^k + \delta u^j\delta\dot{u}^k)\,.\end{aligned} \tag{1.55}$$

In view of (1.54), Eq. (1.55) becomes

$$\begin{aligned}\frac{d}{dt}\delta^2\mathcal{H}_C &= \int d\mu \frac{\delta^2\mathcal{H}_C(\boldsymbol{u}_e)}{\delta u^j\delta u^k}\times \\ &\times\left(\mathcal{J}_e^{jm}\frac{\delta^2\mathcal{H}_C(\boldsymbol{u}_e)}{\delta u^m\delta u^\ell}\delta u^\ell\delta u^k + \mathcal{J}_e^{km}\frac{\delta^2\mathcal{H}_C(\boldsymbol{u}_e)}{\delta u^m\delta u^\ell}\delta u^\ell\delta u^j\right)\,,\end{aligned} \tag{1.56}$$

which is equal to zero due to the antisymmetry of $\mathcal{J}$. Another way to see that $\delta^2\mathcal{H}_C(\boldsymbol{u}_e)$ is a constant of motion for the linearized dynamics is to identify the Hamiltonian in (1.54) by rewriting this equation as

$$\delta\dot{u}^i = \mathcal{J}^{ij}(\boldsymbol{u}_e)\frac{\delta}{\delta u^j}\left(\frac{1}{2}\frac{\delta^2\mathcal{H}_C(\boldsymbol{u}_e)}{\delta u^\ell\delta u^k}\delta u^\ell\delta u^k\right)\,. \tag{1.57}$$

This means that $\frac{1}{2}\delta^2\mathcal{H}_C(\boldsymbol{u}_e)$ is the Hamiltonian of the linearized dynamics, and therefore is a constant of the respective motion. Additionally, if it is definite in sign then either $\delta^2\mathcal{H}_C(\boldsymbol{u}_e)$ or $-\delta^2\mathcal{H}_C(\boldsymbol{u}_e)$ would be positive definite and consequently will provide a conserved norm on the space of perturbations. Therefore, any perturbation with initial energy $\delta^2\mathcal{H}_C(\boldsymbol{u}_e)$ will stay on surface defined by $\delta^2\mathcal{H}_C(\boldsymbol{u}_e) = const.$ and the energy-Casimir equilibrium $\boldsymbol{u}_e$ is Lyapunov stable.

After this analysis we are now in position to summarize the steps to study linear stability using the Casimir invariants arising in noncanonical Hamiltonian systems (see also [10] for an algorithmic presentation of this procedure containing also considerations concerning nonlinear stability):

1. Identify the noncanonical Hamiltonian structure underlying the equations of motion, which requires the identification of a Hamiltonian functional $\mathcal{H}$ and a Poisson bracket $\{\mathcal{F},\mathcal{G}\}$ describing dynamics by $\frac{d\mathcal{F}}{dt} = \{\mathcal{F},\mathcal{H}\}$.

2. From the Casimir-determining equation $\{\mathcal{C},\mathcal{F}\} = 0 \ \forall\mathcal{F}$ determine the complete set of the Casimir invariants $\mathcal{C}$.

3. Construct the energy-Casimir variational principle $\delta(\mathcal{H} + \sum_k \mathcal{C}_k) = 0$ by which equilibrium solutions can be derived.

4. Compute the second order variation of $\mathcal{H}_C = \mathcal{H} + \sum_k \mathcal{C}_k$ at the equilibrium point found in the previous step. Since $\delta^2\mathcal{H}_C(\boldsymbol{u}_e)$ is conserved by the linearized dynamics, its sign definiteness implies linear stability.

1.3.4 Casimir preserving variations - Dynamical accessibility

Different notions for the stability of fluid flows under variations that conserve various properties of the dynamical model were invented over the past decades. For example the so-called "isovortical" variations that conserve the local Kelvin circulation invariants resulting in "equivorticity" flows, were introduced by Arnold [11]. Later on, a similar notion was introduced within the mathematically rigorous framework of noncanonical Hamiltonian formalism by Morrison and Pfirsch in [12]. They applied this method in studying the stability of the Maxwell-Vlasov system under variations called dynamically accessible (DAVs), that are generated by the model dynamics and preserve the Casimir invariants. Also the method was exploited for plasma fluid models e.g. for ideal MHD by Hameiri [13] and later on for Hall MHD in [14] and [15]. The main advantages of this method are: 1) DAVs are generated by the Hamiltonian structure of the system, that provides a systematic algorithm leading to stability criteria, (as was also the case in EC stability method) in view of Dirichlet's theorem, 2) it is applicable to generic equilbria and not only for those resulting from the EC principle and 3) it leads to sufficient stability criteria in cases where the EC method fails due to the lack of Casimirs.

The main idea behind DAVs is that they constrain the phase space trajectory on the symplectic leaves. Such kind of motion can be generated by the noncanonical Poisson bracket using some Hamiltonian that is responsible for the dynamics of the perturbations, usually called generating Hamiltonian functional $\mathcal{W}$,

$$\delta u^i_{da} = \mathcal{J}^{ij}\frac{\delta \mathcal{W}}{\delta u^j} = \{u^i, \mathcal{W}\}\,. \tag{1.58}$$

Evidently these variations are Casimir preserving

$$\delta \mathcal{C}_{da} = \int d\mu \frac{\delta \mathcal{C}}{\delta u^i}\delta u^i_{da} = \int d\mu \frac{\delta \mathcal{C}}{\delta u^j}\mathcal{J}^{ij}\frac{\delta \mathcal{W}}{\delta u^j} = 0\,. \tag{1.59}$$

The second order variation of the Hamiltonian is needed for establishing dynamically accessible stability. Starting with the first order variation we have

$$\delta \mathcal{H}_{da} = \int d\mu \frac{\delta \mathcal{H}}{\delta u^i}\delta u^i_{da} = \int d\mu \frac{\delta \mathcal{H}}{\delta u^i}\mathcal{J}^{ij}\frac{\delta \mathcal{W}}{\delta u^j}\,, \tag{1.60}$$

which vanishes at $\boldsymbol{u} = \boldsymbol{u}_e$. Therefore condition $\{\mathcal{H}, \mathcal{W}\} = 0$ provides us with the equilibrium equations. Furthermore, upon considering the second order variation of $\mathcal{H}$ we take

$$\delta^2 \mathcal{H}_{da} = \int d\mu \left(\frac{1}{2}\frac{\delta^2 \mathcal{H}}{\delta u^i \delta u^j}\delta u^i_{da}\delta u^j_{da} + \frac{\delta \mathcal{H}}{\delta u^i}\delta^2 u^i_{da}\right)\,, \tag{1.61}$$

where

$$\begin{aligned} \delta^2 u^i_{da} = \{u^i, \mathcal{W}^{(2)}\} + \frac{1}{2}\{\{u^i, \mathcal{W}^{(1)}\}, \mathcal{W}^{(1)}\} = \\ \mathcal{J}^{ij}\frac{\delta \mathcal{W}^{(2)}}{\delta u^j} + \frac{1}{2}\left(\mathcal{J}^{j\ell}\frac{\partial \mathcal{J}^{ik}}{\partial u^j}\frac{\delta \mathcal{W}^{(1)}}{\delta u^k}\frac{\delta \mathcal{W}^{(1)}}{\delta u^\ell} + \mathcal{J}^{ij}\mathcal{J}^{\ell k}\frac{\delta^2 \mathcal{W}^{(1)}}{\delta u^j \delta u^\ell}\frac{\delta \mathcal{W}^{(1)}}{\delta u^k}\right)\,. \end{aligned} \tag{1.62}$$

Here $\mathcal{W}^{(2)}$ is a second order generating functional, which is irrelevant though in computing $\delta^2 \mathcal{H}_{da}$ because when the second term in the rhs of (1.61) is multiplied with the first term in the rhs of (1.62), vanishes in view of the equilibrium condition. Eventually one has

$$\begin{aligned} \delta^2 \mathcal{H}_{da}[\boldsymbol{u}_e; \delta \boldsymbol{u}_{da}] = \frac{1}{2}\int d\mu \bigg(\frac{\delta^2 \mathcal{H}}{\delta u^i \delta u^j}\mathcal{J}^{i\ell}\mathcal{J}^{jk}\frac{\delta \mathcal{W}^{(1)}}{\delta u^\ell}\frac{\delta \mathcal{W}^{(1)}}{\delta u^k} \\ + \frac{\delta \mathcal{H}}{\delta u^i}\mathcal{J}^{j\ell}\frac{\partial \mathcal{J}^{ik}}{\partial u^j}\frac{\delta \mathcal{W}^{(1)}}{\delta u^k}\frac{\delta \mathcal{W}^{(1)}}{\delta u^\ell} + \frac{\delta \mathcal{H}}{\delta u^i}\mathcal{J}^{ij}\mathcal{J}^{\ell k}\frac{\delta^2 \mathcal{W}^{(1)}}{\delta u^j \delta u^\ell}\frac{\delta \mathcal{W}^{(1)}}{\delta u^k}\bigg)\,, \end{aligned} \tag{1.63}$$

where the Poisson operator and the variational derivatives of the Hamiltonian are computed at the equilibrium point. Usually the functional generating the DAVs is formed as follows

$$\mathcal{W} = \int d\mu\, u^i \mathrm{g}_i\,, \tag{1.64}$$

with g_i being components of arbitrary vectors which encapsulate the arbitrariness of the variations. Notice that since the variables u^i are involved linearly in $\mathcal{W}^{(1)}$ the last term in (1.63) vanishes. After these calculations a sufficient stability criterion can be readily established: *An equilibrium point $\boldsymbol{u}_e$ is stable if $\delta^2\mathcal{H}_{da}$ is definite in sign.*

1.4 Lagrangian stability

For the description of fluids there are two different viewpoints which result in canonical and noncanonical Hamiltonian formalisms, respectively: the Lagrangian picture and the Eulerian picture. The Lagrangian formulation describes the fluid as consisting of many small fluid elements and tracks the motion of every single element during the dynamical evolution. Within the Eulerian viewpoint the fluid motion is described by means of fields, both scalar and vector ones, measured at fixed point in space; as a result this latter viewpoint is natural for describing fluid motion in the observer's frame. So, the equations of motion for fluid dynamics are usually expressed in Eulerian variables. However, the Lagrangian picture provides a more natural framework for action and variational principles, since the generalization of the well known action principles to infinite dimensions is straightforward within the well established canonical Lagrangian/Hamiltonian description of dynamics. Lagrangian description provides a quite general method for stability analysis, since there are no geometric or dynamical restrictions on the permissible perturbations like in the two methods described previously.

In the Lagrangian framework the fluids are described in terms of Lagrangian or material variables suitable for tracking the motion of the individual fluid elements. The material variables are the positions of the fluid elements at a given instant $\mathbf{q}_s(\mathbf{a},t)$ ($s=i,e$ standing for the ion and electron species) where $\mathbf{a}\in\mathbb{R}^3$ is the fluid element label, usually taken as the element's position at $t=0$. The two viewpoints are connected through the so-called Lagrange-Euler map, which has to be consistent in the sense that an action written in the Lagrangian framework is mapped to an action written exclusively in terms of Eulerian variables, a requirement called Eulerian Closure Principle (ECP) [16, 17]. For an ideal fluid theory, the fluid part of the Eulerian dynamics is described by n, $\mathbf{v}$, s, (here s is the specific entropy) and the Lagrange-Euler map, presented in detail in [8], is given by the following relations

$$n(\mathbf{x},t)=\left.\frac{n_0(\mathbf{a})}{\mathcal{J}(\mathbf{a},t)}\right|_{\mathbf{a}=\mathbf{q}^{-1}(\mathbf{x},t)}, \tag{1.65}$$

$$\mathbf{v}(x,t)=\left.\dot{\mathbf{q}}(\mathbf{a},t)\right|_{\mathbf{a}=\mathbf{q}^{-1}(\mathbf{x},t)}, \tag{1.66}$$

$$s(\mathbf{x},t)=\left.s_0(\mathbf{a})\right|_{\mathbf{a}=\mathbf{q}^{-1}(\mathbf{x},t)}, \tag{1.67}$$

where $\mathcal{J}(\mathbf{a},t) = \det(\partial q^i/\partial a^j)$.

A standard approach to perform stability analysis within this framework is to write the Hamiltonian of our ideal fluid theory in terms of material variables $\mathbf{q}$, $\boldsymbol{\pi}$. To do so we construct the Lagrangian functional describing fluid dynamics in material variables. Then one has to expand the Lagrangian up to second order upon considering small perturbations around a reference trajectory i.e.

$$\mathbf{q}(\mathbf{a},t) = \mathbf{Q}(\mathbf{a},t) + \boldsymbol{\zeta}(\mathbf{a},t)\,, \tag{1.68}$$

where ζ is the so-called Lagrangian displacement vector. The reference trajectory need not to correspond to a Lagrangian equilibrium. Actually for a flowing Eulerian equilibrium the reference state is necessarily time dependent in the Lagrangian framework because from (1.67) we understand that if we consider a Lagrangian equilibrium trajectory, i.e. $\dot{\mathbf{q}} = 0$, then the Eulerian velocity will vanish $\mathbf{v}(\mathbf{x},t) = 0$. In view of (1.68) we expand the Lagrangian as follows

$$\mathcal{L} = \mathcal{L}_0 + \delta\mathcal{L}_{la} + \delta^2\mathcal{L}_{la} + \ldots\,, \tag{1.69}$$

where $\mathcal{L}_0$ is a constant, $\delta\mathcal{L}_{la}$ vanishes at equilibrium and $\delta^2\mathcal{L}_{la}$ governs the linearized dynamics. Applying the Lagrange-Euler map one has to find a Lagrangian completely expressible in terms of Eulerian variables which moreover produces the correct Euler-Lagrange equations. Then from the Eulerian counterpart of $\delta^2\mathcal{L}_{la}$ one can construct the Hamiltonian $\delta^2\mathcal{H}$ upon employing a Legendre transform. Then the infinite dimensional version of Dirichlet's stability theorem provides the sufficient stability condition $\delta^2\mathcal{H} > 0$.

1.5 Hamiltonian description of XMHD

It has been recognized recently that the dynamical equations of barotropic XMHD in Eulerian description, i.e. (1.27), (1.29) and (1.19), possess a noncanonical Hamiltonian structure [18]. According to the previous section, this means that the dynamics can be described by a set of generalized Hamiltonian equations

$$\partial_t u = \{u, \mathcal{H}\}\,, \tag{1.70}$$

where u is a member of $(\rho, \mathbf{v}, \mathbf{B}^*)$, which are noncanonical dynamical variables (they do not form canonical conjugate pairs), $\mathcal{H}[\rho, \mathbf{v}, \mathbf{B}^*]$ is a real valued Hamiltonian functional, and $\{F, G\}$ is a noncanonical Poisson bracket (see [8, 19]). which is bilinear, antisymmetric, and satisfies the Jacobi identity. The appropriate Hamiltonian for our system is the following:

$$\begin{aligned}\mathcal{H} &= \frac{1}{2}\int_D d^3x \left[\rho|\mathbf{v}|^2 + 2\rho U(\rho) + B^2 + d_e^2\frac{|\nabla\times\mathbf{B}|^2}{\rho}\right] ,\\ &= \frac{1}{2}\int_D d^3x \left[\rho|\mathbf{v}|^2 + 2\rho U(\rho) + \mathbf{B}\cdot\mathbf{B}^*\right] ,\end{aligned} \tag{1.71}$$

where $D \subseteq \mathbb{R}^3$ and U is the internal energy function ($p = \rho^2 dU/d\rho$), while the corresponding noncanonical Poisson bracket is

$$\begin{aligned}\{F,G\} = \int_D d^3x \, \{ & G_\rho\boldsymbol{\nabla}\cdot F_{\mathbf{v}} - F_\rho\boldsymbol{\nabla}\cdot G_{\mathbf{v}} + \rho^{-1}(\boldsymbol{\nabla}\times\mathbf{v})\cdot(F_{\mathbf{v}}\times G_{\mathbf{v}}) \\ & +\rho^{-1}\mathbf{B}^*\cdot[F_{\mathbf{v}}\times(\boldsymbol{\nabla}\times G_{\mathbf{B}^*}) - G_{\mathbf{v}}\times(\boldsymbol{\nabla}\times F_{\mathbf{B}^*})] \\ & -d_i\rho^{-1}\mathbf{B}^*\cdot[(\boldsymbol{\nabla}\times F_{\mathbf{B}^*})\times(\boldsymbol{\nabla}\times G_{\mathbf{B}^*})] \\ & +d_e^2\rho^{-1}(\boldsymbol{\nabla}\times\mathbf{v})\cdot[(\boldsymbol{\nabla}\times F_{\mathbf{B}^*})\times(\boldsymbol{\nabla}\times G_{\mathbf{B}^*})]\} ,\end{aligned} \tag{1.72}$$

where $F_{\boldsymbol{u}} := \delta F/\delta\boldsymbol{u}$ denotes the functional derivative of F with respect to the dynamical variable $\boldsymbol{u}$, defined by $\delta F[\boldsymbol{u},\delta\boldsymbol{u}] = \int_D d^3x\,\delta\boldsymbol{u}\cdot(\delta F/\delta\boldsymbol{u})$. For the computation of the functional derivatives of the field variables we make use of $\delta u_i(\boldsymbol{x}')/\delta u_j(\boldsymbol{x}) = \delta_{ij}\delta(\boldsymbol{x}'-\boldsymbol{x})$.

For the general 3D version of the model described by means of (1.71) and (1.72), the Casimir invariants i.e. the functionals $\mathcal{C}$ that satisfy $\{\mathcal{C},\mathcal{F}\} = 0 \;\forall\mathcal{F}$, are

$$\mathcal{C}_1 = \int_D d^3x\,\rho , \tag{1.73}$$

$$\mathcal{C}_{2,3} = \int_D d^3x\,(\mathbf{A}^* + \gamma_\pm\mathbf{v})\cdot(\mathbf{B}^* + \gamma_\pm\boldsymbol{\nabla}\times\mathbf{v}) , \tag{1.74}$$

with $\mathbf{B}^* = \boldsymbol{\nabla}\times\mathbf{A}^*$ and $\gamma_\pm$ being the two roots of the quadratic equation $\gamma^2 - d_i\gamma - d_e^2 = 0$, i.e. $\gamma_\pm = \left(d_i \pm \sqrt{d_i^2 + 4d_e^2}\right)/2$.

Remarkably, in [20] the authors derived (1.72) starting from a Lagrangian-Action formulation and taking the Lagrange-Euler map of the canonical Poisson bracket. Also, in [21] the authors discovered an interesting property of (1.72), that is, if the bracket is expressed in terms of generalized vorticity fields given by

$$\mathbf{B}^\pm = \mathbf{B}^* + \gamma_\pm\nabla\times\mathbf{v} , \tag{1.75}$$

then it assumes the form of the Hall MHD Poisson bracket with $d_i \to d_i - 2\gamma_\pm$, i.e. the following identity holds

$$\{F,G\}_{xmhd}[d_i, d_e^2; \mathbf{B}^*] = \{F,G\}_{hmhd}[\nu_\pm; \mathbf{B}^\pm] , \tag{1.76}$$

where $\nu_\pm = d_i - 2\gamma_\pm$. This result is exploited in this thesis in order to obtain simplifications of the spatially reduced brackets that facilitate the computation of the corresponding Casimir invariants.

1.6 Motivation and aim

The study of equilibrium and stability of plasmas is very important for fusion and astrophysical research. Especially for fusion applications, equilibrium and stability are crucial factors for the attainment of long lived states in magnetic confinement devices, such as the Tokamak and the Stellarator, with sufficient confinement of thermal energy for the self-sustained operation of thermonuclear reactors. In general the most drastic way to lose the confinement of plasma energy is the development of either macro-instabilities, e.g. the current driven kink and the pressure driven ballooning instability associated with plasma disruption (effectively they put upper limits on the attainable current and pressure), or micro-instabilities that result in enhanced turbulence and anomalous transport. Stability analyses are usually performed using the standard MHD energy principle [22] that was generalized for flowing equilbria in [23]. The equilibrium and stability analysis of stationary plasma states with macroscopic sheared flows, albeit a tough problem from the mathematical point of view, is important since it is believed that plasma rotation, either being self-generated or driven externally, may have beneficial effects in terms of confinement. Indeed plasma flows are associated with the suppression of turbulence [24] and the L-H transitions [2] observed in Tokamaks. Also there are many studies proposing that plasma sheared rotation variously affects the stability properties of Tokamak equilibria in several cases, either inducing stabilization or destabilization (e.g. [25, 26, 27, 28, 29]), with the main destabilizing mechanism being the Kelvin-Helmholtz instability [30].

Furthermore, many astrophysical phenomena, such as the development of turbulence in various stages of the solar wind and in magnetized accretion disks, are consequences of flow-driven instabilities, such as the Kelvin-Helmholtz (e.g. see [31]) and the Magneto-rotational instability (MRI) [32]. It is evident that plasma instability is the reason for the emergence of new structures but most importantly for fusion physics, they are also the main mechanisms behind the undesirable interchange of energy, which should be sufficiently reduced in fusion experiments. This pursuit is the main reason for performing stability studies for over sixty years, trying to refine the resulting stability or instability criteria and incorporate as many physics as possible.

It is widely agreed that ordinary MHD, despite being a successful model for describing macroscopic phenomena, provides a rather rough description of plasmas since it neglects the presence of multi-fluid components. This is especially true when there exist characteristic length scales comparable to the ion and electron skin depths, e.g., due to the presence of current sheets or thin boundary layers. In such cases multi-fluid models are needed to describe phenomena arising due to the coexistence of different particle species and the decoupling of their respective motions, even on macroscopic level. Regarding stability, when mode frequencies comparable to the particle gyrofrequencies are present then MHD becomes clearly an insufficient framework. This intuitive reasoning about the insufficiency of the MHD model, is corroborated when MHD theory fails to predict adequately the experimental observations: the observed stability of elongated Field Reversed Configurations (FRC) [33, 34] and the high magnetic reconnection rates (for example see [35, 36]) are examples where two-fluid models work significantly better than MHD. Moreover, there exist recent views on Tokamak physics suggesting that the Hall drift term cannot be neglected both in equilibrium and dynamics computations; also it has been suggested that Hall effects may be associated with the pressure pedestals, formed in the L-H transitions [37, 38].

For the reasons described above, very often we need to invoke multi-fluid descriptions since they capture finer dynamical effects, taking place in shorter length and temporal scales. Regarding stability analysis of flowing plasmas though, a two-fluid treatment is an even tougher problem. If rotation is neglected the two fluid effects are incorporated more easily through the multi-fluid pressure (e.g. see [14]) because no decoupling of electron and ion motion occurs. However, as was stressed earlier, plasma flows are consequential and therefore it is important to take them into account. A characteristic consequence of including flows in stability methods based on energy functionals is the non-separability of the kinetic and potential energy contributions rendering the resulting stability criteria sufficient but not necessary. A typical example is the MHD energy principle, which for static equilibria provides a necessary and sufficient condition [22], while for stationary states [23] it provides only sufficient conditions. These are, respectively, the Lagrange and Dirichlet conditions of Hamiltonian dynamics, as pointed out in [8].

In this thesis we deal with formal stability. By this term we mean an analysis based on an energy-like quantity that is conserved by the full nonlinear dynamics of the system. For formal stability, the first variation of this quantity must vanish and the second variation must be positive (or negative) definite at the equilibrium. When this is the case, the second variation serves as a Lyapunov functional for linear dynamics. Formal stability is important because it implies linearized and spectral stability and is a step forward to nonlinear stability that requires additional convexity estimates [10, 8]. Thus far, only a limited number of studies have led to appropriate Lyapunov functionals and ultimately to rigorous conclusions within the two-fluid context, primarily in the Hall MHD (HMHD) limit [39, 40, 14, 41, 15], and a few of them employing the complete two-fluid model [42, 43].

A very useful apparatus for conducting equilibrium and especially stability studies is the Hamiltonian description of the ideal fluids. Undoubtedly, the identification of a

Hamiltonian structure underlying the dynamics of ideal MHD [44] was of fundamental importance for the utilization and the correct interpretation of several variational methods that had been used for equilibrium and stability since the early days of theoretical plasma physics, e.g., [45, 46, 47, 48]. These variational principles were introduced in an ad hoc manner, based on physical arguing and conjectures. It was only with the development of the noncanonical Hamiltonian theory, that a rigorous mathematical justification was attributed to these methods. Also this formalism facilitated stability analyses due to the fact that a Hamiltonian description directly suggests good candidates for Lyapunov functionals, whose very existence is closely related with the noncanonical nature of the dynamics e.g. the second variation of the EC functional. In particular the knowledge of the MHD Poisson bracket enables a correct and straightforward derivation of the dynamically accessible (DA) stability criterion [13, 49]. Some of these advantages of the Hamiltonian formulation were also exploited for Hall MHD (HMHD) but even less transfer of methodologies took place for two-fluid models containing also electron inertial physics.

XMHD is perhaps the simplest consistent, in terms of energy conservation [50], fluid plasma model containing both Hall drift and electron inertial effects. These effects, arising due to the existence of at least two particle species consisting the plasma, are neglected by the well known and widely employed ideal MHD model. Although it was introduced a long time ago [6], its complicated general form was a limiting factor for the identification of its Hamiltonian structure. Recently this identification came into existence in [18] for the barotropic version of the model and it was corroborated in [21], where similarities with the Hamiltonian structures of HMHD (e.g. [51]), Inertial MHD (IMHD) [50, 52] and ordinary MHD, were identified. In addition the Hamiltonian structure of XMHD served as the starting point for a subsequent paper that dealt with the application of its translationally symmetric counterpart to magnetic reconnection [53].

The facts and the developments described above were the driving force behind the conception and the realization of the present study, since with the noncanonical Hamiltonian structure of the model at hand, Hamiltonian variational principles can be employed both for the study of equilibrium and stability within the framework of a model able to describe much more physical processes than ordinary MHD. More specifically, in the present thesis the Hamiltonian formulation of the barotropic XMHD model with continuous helical symmetry, a general case of continuous spatial symmetry that contains both the cases of axial and translation symmetry, is presented. A helically symmetric formulation can be easily reduced to an axially symmetric one that enables the description of the Tokamak plasmas and Reversed Field Configurations or astrophysical plasmas e.g. Pulsar magnetospheres. On the other hand, helically symmetric formulation itself is interesting because purely or nearly helical structures are very common in plasma systems. For example, 3D equilibrium states with internal helical structures, e.g., helical cores, have been observed experimentally [54, 55] and simulated [56, 57] in Tokamaks and RFPs (e.g. [58, 59, 60, 61]). Also, helical structures emerge due to plasma instabilities, such as the resistive or collisionless tearing modes, or as a result of externally imposed symmetry-breaking perturbations are magnetic islands [62]. In addition the helix may serve as a rough approximation of helical non-axisymmetric devices [63] and can be useful to investigate some features of stellarators [64, 65], the second major class of magnetic confinement devices alongside the Tokamak, in the large aspect-ratio limit. Also, helical magnetic structures are common in astrophysics, e.g., in astrophysical jets [66, 67]. Therefore, it is of interest to derive a joint tool for two-fluid equilibrium and stability studies of systems with helical symmetry, with the understanding that for most cases of laboratory application helical symmetry is an idealized approximation to the large aspect ratio limit of the above mentioned toroidal systems.

Aim of this thesis is to transfer the Hamiltonian variational methods for evaluating equilibrium and stability to the uncharted waters of XMHD and also to a variety of two-fluid models with quasineutrality, such as the complete quasineutral two fluid model, Hall MHD and Inertial MHD, staying though conceptually and formalistically as close as possible to MHD. Given the historical precedent, it would appear desirable to employ equilibrium and stability analysis methods similar to those originating from the MHD Grad-Shafranov-Bernoulli equilibrium system and the MHD energy principle, respectively, because this framework is already well known and also because this would facilitate comparisons with the MHD results.

Another aim is to reveal some of the consequences stemming from the introduction of the two-fluid contributions with the help of certain instructive applications and also to highlight and provide solutions to the formalistic challenges emerging when working within this framework. For example, the derivation of the helically symmetric Poisson bracket and of the ellipticity condition for the axisymmetric XMHD equilibrium equations, and also the computation of second order variations of the Hamiltonian functionals, under DA and Lagrangian perturbations, become transparent. The methodologies, developed to address these challenges, can be exploited in other studies where similar models are used.

1.7 Thesis outline

The results of this thesis are presented in four main chapters, namely

1. Hamiltonian formulation of symmetric XMHD (Chapter 2).

2. Extended MHD equilibria (Chapter 3).

3. Stability analysis of XMHD equilibria (Chapter 4).

4. Alternative bracket formulations of incompressible XMHD (Chapter 5).

More specifically, in Chapter 2 the helically symmetric, noncanonical Poisson bracket is derived by spatially reducing the three dimensional one in view of a symmetric representation of the field variables. In addition, the Casimir invariants corresponding to this bracket are computed upon solving the Casimir determining equations and their Hall MHD and MHD limits are obtained. In Chapter 3 the EC variational principle is employed to obtain equilbrium equations describing generic three dimensional, helically symmetric and axially symmetric equilibria. Also special cases are discussed and two particular applications are presented. The first concerns the numerical solution of the axisymmetric Hall MHD equilibrium equations and the second concerns the derivation of an analytic double-Beltrami solution for the helically symmetric incompressible HMHD. Moreover, the ellipticity condition for the axisymmetric, barotropic XMHD equilibrium equations is derived and special cases are discussed. Chapter 4 deals with the stability analysis of XMHD equilibria where the EC, the DA and the Lagrangian stability methods are exploited. An application of a special EC stability criterion for equilibria with toroidal rotation is also presented. Moreover, in the framework of Lagrangian stability, the special case of HMHD is thoroughly worked out. In Chapter 5 an alternative formulation of XMHD dynamics is proposed in terms of trilinear brackets. In addition, a heuristic method for the construction of the model equations from the conservation laws is proposed using the simpler reduced MHD (RMHD) model. In Chapter 6 the results contained in the aforementioned chapters are summarized and the conclusions are presented. Also we propose future research plans and topics which could potentially emerge from this thesis.

Chapter 2

Hamiltonian formulation of helically symmetric XMHD

This is the first chapter that contains original results, published in [68]. It is devoted to the derivation of the helically symmetric formulation of XMHD dynamics, and the corresponding Poisson bracket and Casimir invariants.

The aforementioned results are presented in two sections: in Section 2.1 the helically symmetric Poisson bracket is derived, providing the helically symmetric XMHD dynamical equations. Also, the transformation (1.76) is performed for the helically symmetric bracket. In Section 2.2 we exploit the transformed bracket to compute the helically symmetric Casimir invariants. In addition the HMHD, IMHD and MHD limits of the Casimirs are obtained.

2.1 Helical symmetry and Poisson bracket reduction

Helical symmetry can be imposed by assuming that in a cylindrical coordinate system (r, ϕ, z) all quantities and equations of motion depend spatially on r and on the helical coordinate $u = \ell\phi + nz$, where $\ell = \sin(a)$ and $n = -\cos(a)$, with a being the helical angle. For $a = 0$ and $a = \pi/2$ we obtain the axisymmetric and the translationally symmetric cases, respectively. The contravariant unit vector in the direction of the u coordinate is

$$\boldsymbol{e}_u = \frac{\nabla u}{|\nabla u|} = \ell k \boldsymbol{e}_\phi + nkr\boldsymbol{e}_z \,, \tag{2.1}$$

where k is

$$k := \frac{1}{\sqrt{\ell^2 + n^2 r^2}} \,. \tag{2.2}$$

The tangent to the direction of the helix $r = const.$ $u = const.$ is given by

$$\mathbf{e}_h = \mathbf{e}_r \times \mathbf{e}_u \,, \tag{2.3}$$

and one can prove that the following relations hold:

$$\nabla \cdot \mathbf{h} = 0\,, \quad \nabla \times \mathbf{h} = -2n\ell k^2 \mathbf{h}\,, \tag{2.4}$$

where

$$\mathbf{h} = k\mathbf{e}_h = \frac{\ell \nabla z - n r^2 \nabla \phi}{\ell^2 + n^2 r^2}\,, \tag{2.5}$$

with $\mathbf{h}\cdot\mathbf{h} = k^2$. Helical symmetry means that $\mathbf{h}\cdot\nabla f = 0$, where f is an arbitrary scalar function. Relations (2.4) enable us to introduce the so-called poloidal representation for the divergence-free magnetic field and also a poloidal representation for the velocity field with a potential field contribution accounting for the compressibility of the flow, i.e.,

$$\mathbf{B}^* = k^{-1} B_h^*(r,u,t)\mathbf{h} + \nabla\psi^*(r,u,t) \times \mathbf{h}\,, \tag{2.6}$$

$$\mathbf{v} = k^{-1} v_h(r,u,t)\mathbf{h} + \nabla\chi(r,u,t) \times \mathbf{h} + \nabla\Upsilon(r,u,t)\,. \tag{2.7}$$

For incompressible flows Υ is harmonic or constant. In view of (2.4), the divergence and the curl of (2.6) and (2.7) are given by

$$\nabla \cdot \mathbf{v} = \Delta\Upsilon\,, \quad \nabla \cdot \mathbf{B}^* = 0\,, \tag{2.8}$$

$$\nabla \times \mathbf{v} = \left[k^{-2}\mathcal{L}\chi - 2n\ell k v_h\right]\mathbf{h} + \nabla(k^{-1} v_h) \times \mathbf{h}\,, \tag{2.9}$$

$$\nabla \times \mathbf{B}^* = \left[k^{-2}\mathcal{L}\psi^* - 2n\ell k B_h^*\right]\mathbf{h} + \nabla(k^{-1} B_h^*) \times \mathbf{h}\,, \tag{2.10}$$

where $\Delta := \nabla^2$ and $\mathcal{L} := -\nabla \cdot (k^2 \nabla(\cdot))$ is a linear, self-adjoint differential operator. For convenience we define the following quantities: $w := \Delta\Upsilon$ or $\Upsilon = \Delta^{-1} w$ and $\Omega = \mathcal{L}\chi$ or $\chi = \mathcal{L}^{-1}\Omega$.

Having introduced representation (2.6)–(2.7) for the helically symmetric fields, in order to derive the helically symmetric Hamiltonian formulation we need to express the Hamiltonian (1.71) and the Poisson bracket (1.72) in terms of the scalar field variables $\boldsymbol{u}_{hs} = (\rho, v_h, \chi, \Upsilon, B_h^*, \psi^*)$. This can be accomplished upon expressing the fields $\boldsymbol{u}_{3D} = (\rho, \mathbf{v}, \mathbf{B}^*)$ in terms of the scalar field variables and also upon transforming the functional derivatives with respect to $\boldsymbol{u}_{3D}$ to functional derivatives with respect to the scalar fields $\boldsymbol{u}_{HS}$. As in [69, 70, 71], we perform this transformation by employing a chain rule reduction. The chain rule for functional derivatives (see [69]) is obtained by equating the first variations of arbitrary functionals in terms of the 3D variables to the corresponding variations in terms of the helically symmetric variables. The variation of a functional $F[\rho, \mathbf{v}, \mathbf{B}^*]$ is

$$\delta F[\boldsymbol{u}_{3D}] = \int_V d^3x \, \left(F_\rho \delta\rho + F_{\mathbf{v}} \cdot \delta\mathbf{v} + F_{\mathbf{B}^*} \cdot \delta\mathbf{B}^*\right)\,, \tag{2.11}$$

while that in terms of helically symmetric variables is

$$\begin{aligned}\delta F[\boldsymbol{u}_{hs}] &= \int_D d^2x \Big[F_\rho \delta\rho + F_{v_h}\delta v_h + F_\chi \delta\chi + F_\Upsilon \delta\Upsilon \\ &\qquad + F_{B_h^*}\delta B_h^* + F_{\psi^*}\delta\psi^*\Big] ,\end{aligned} \tag{2.12}$$

where $D \subseteq R^2$ is a restriction of V to R^2. It is evident that $\delta v_h = k^{-1}\mathbf{h}\cdot\delta\mathbf{v}$, $\delta B_h^* = k^{-1}\mathbf{h}\cdot\delta\mathbf{B}^*$ and $\delta\Upsilon = \Delta^{-1}\delta w = \Delta^{-1}(\nabla\cdot\delta\mathbf{v})$. Also it is not difficult to see, upon taking the vector product of (2.6) with $\mathbf{h}$, that $\delta\psi^* = -\Delta^{-1}\left[\nabla\left(k^{-2}\delta\mathbf{B}\times\mathbf{h}\right)\right]$. Finally, in order to find the relation connecting $\delta\chi$ to $\delta\mathbf{v}$ we project the vorticity along $\mathbf{h}$ to obtain

$$\mathbf{h}\cdot\nabla\times\mathbf{v} = \mathcal{L}(\chi) - 2n\ell k^2\mathbf{v}\cdot\mathbf{h} , \tag{2.13}$$

hence, one has $\delta\chi = \mathcal{L}^{-1}(\mathbf{h}\cdot\nabla\times\delta\mathbf{v}) + 2n\ell\mathcal{L}^{-1}\left(k^{-2}\mathbf{h}\cdot\delta\mathbf{v}\right)$. In view of these relations (2.12) can be rewritten as

$$\begin{aligned}\delta F[\boldsymbol{u}_{hs}] &= \int_D d^2x \Big\{F_\rho\delta\rho + k^{-1}F_{v_h}\mathbf{h}\cdot\delta\mathbf{v} + F_\chi\mathcal{L}^{-1}\left(\mathbf{h}\cdot\nabla\times\delta\mathbf{v} + 2n\ell k^2\mathbf{h}\cdot\delta\mathbf{v}\right) \\ &+ F_\Upsilon\Delta^{-1}(\nabla\cdot\delta\mathbf{v}) + k^{-1}F_{B_h^*}\mathbf{h}\cdot\delta\mathbf{B}^* - F_{\psi^*}\Delta^{-1}\left[\nabla\left(k^{-1}\delta\mathbf{B}^*\times\mathbf{h}\right)\right]\Big\} .\end{aligned} \tag{2.14}$$

Then, from the self-adjointness of the operators Δ^{-1} and $\mathcal{L}^{-1}$ and for appropriate boundary conditions, such that the boundary terms arising from integrations by parts vanish, we obtain

$$\delta F = \int_D d^2x\,(F_\rho\delta\rho + F_{\mathbf{v}}\cdot\delta\mathbf{v} + F_{\mathbf{B}^*}\cdot\delta\mathbf{B}^*) \tag{2.15}$$

$$\begin{aligned}&= \int_D d^2x\,\Big\{F_\rho\delta\rho + \left[k^{-1}F_{v_h}\mathbf{h} + \nabla\mathcal{L}^{-1}F_\chi\times\mathbf{h} - \nabla\left(\Delta^{-1}F_\Upsilon\right)\right]\cdot\delta\mathbf{v} \\ &\quad + \Big[k^{-1}F_{B_h^*}\mathbf{h} - k^{-2}\nabla\left(\Delta^{-1}F_{\psi^*}\right)\times\mathbf{h}\Big]\cdot\delta\mathbf{B}^*\Big\} .\end{aligned} \tag{2.16}$$

Thus, for arbitrary variations the following relations can be deduced

$$F_\rho = F_\rho\,, \quad F_{\mathbf{v}} = k^{-1}F_{v_h}\mathbf{h} + \nabla F_\Omega\times\mathbf{h} - \nabla F_w\,, \tag{2.17}$$

$$F_{\mathbf{B}^*} = k^{-1}F_{B_h^*}\mathbf{h} - k^{-2}\nabla\left(\Delta^{-1}F_{\psi^*}\right)\times\mathbf{h}\,, \tag{2.18}$$

where

$$F_w = \Delta^{-1}F_\Upsilon\,, \qquad F_\Omega = \mathcal{L}^{-1}F_\chi\,, \tag{2.19}$$

which follow from

$$\int_D d^3x\,F_\chi\delta\chi = \int_D d^3\,xF_\Omega\delta\Omega\,, \tag{2.20}$$

$$\int_D d^3x\, F_\Upsilon \delta\Upsilon = \int_D d^3x\, F_w \delta w\,, \tag{2.21}$$

upon introducing the relations $\delta\Omega = \mathcal{L}\delta\chi$, $\delta w = \Delta\delta\Upsilon$ and exploiting the self-adjointness of the operators Δ and $\mathcal{L}$. Also, we observe that in (1.72) there exist bracket blocks which contain the curl of $F_{\mathbf{B}^*}$, which is

$$\nabla\times F_{\mathbf{B}^*} = \left(k^{-2}F_{\psi^*} - 2n\ell k F_{B_h^*}\right)\mathbf{h} + \nabla\left(k^{-1}F_{B_h^*}\right)\times\mathbf{h}\,. \tag{2.22}$$

The helically symmetric Poisson bracket is found by substituting (2.6), (2.9), (2.17) and (2.22) into (1.72) and assuming that any surface-boundary terms which emerge due to integrations by parts, vanish due to appropriate boundary conditions, for example periodic conditions or for field variables u_{HS} vanishing on ∂D, except for the mass density ρ, which has to be finite on the boundary, or to vanish approaching zero slower than the other quantities, otherwise various terms would diverge, as is evident even from (1.72). Let us now see how the various terms in (1.72) are reduced to their helically symmetric counterparts starting with the compressional part

$$\{X^1, X^2\}_{comp} = \epsilon_{jm}\int d^3x\, X^m_\rho \nabla\cdot X^j_{\mathbf{v}} = \epsilon_{jm}\int d^3x\, X^j_\rho X^m_\Upsilon\,, \tag{2.23}$$

where ϵ_{jm} is the antisymmetric permutation symbol and $X^{1,2}$ are arbitrary functionals e.g. $X^1 = F$, $X^2 = G$ (the superscript is merely an index). Note that we have used $\nabla\cdot F_{\mathbf{v}} = -F_\Upsilon$ that can be easily deduced by (2.17). For the vortical part Eqs. (2.9) and (2.17) have to be invoked

$$\{X^1, X^2\}_{vort} = \frac{\epsilon_{jm}}{2}\int d^3x \rho^{-1}(\nabla\times\mathbf{v})\cdot\left(X^j_{\mathbf{v}}\times X^m_{\mathbf{v}}\right)\,. \tag{2.24}$$

Employing straightforward vector analysis manipulations we can show that

$$\begin{aligned}
&\frac{\epsilon_{jm}}{2}\left(X^j_{\mathbf{v}}\times X^m_{\mathbf{v}}\right) = \\
&= \frac{\epsilon_{jm}}{2}\left(k^{-1}X^j_{v_h}\mathbf{h} + \nabla X^j_\Omega\times\mathbf{h} - \nabla X^j_w\right)\times\left(k^{-1}X^m_{v_h}\mathbf{h} + \nabla X^m_\Omega\times\mathbf{h} - \nabla X^m_w\right)\\
&= \epsilon_{jm}\Big\{kX^j_{v_h}\nabla X^m_\Omega - k^{-1}X^j_{v_h}\mathbf{h}\times\nabla X^m_w + \frac{1}{2}[X^j_\Omega, X^m_\Omega]\mathbf{h}\\
&\qquad + \frac{1}{2}\nabla X^j_w\times\nabla X^m_w + (\nabla X^j_w\cdot\nabla X^m_\Omega)\mathbf{h}\Big\}\,.
\end{aligned} \tag{2.25}$$

where $[f,g] := (\nabla f\times\nabla g)\cdot\mathbf{h}$ is the helical Jacobi-Poisson bracket. Taking the scalar product of (2.25) with $\nabla\times\mathbf{v} = \left(k^{-2}\Omega - 2n\ell k v_h\right)\mathbf{h} + \nabla(k^{-1}v_h)\times\mathbf{h}$ we find that the vortical part of $\{X^1, X^2\}$ is given by

$$\epsilon_{jm}\int d^3x\rho^{-1}\Big\{k^2\left(k^{-2}\Omega - 2n\ell k v_h\right)\left(\frac{1}{2}[X^j_\Omega, X^m_\Omega] + \nabla X^j_w\cdot\nabla X^m_\Omega + \frac{1}{2k^2}[X^j_w, X^m_w]\right)$$

$$+kX^j_{v_h}\left([X^m_\Omega, k^{-1}v_h] + \nabla(k^{-1}v_h)\cdot\nabla X^m_w\right)\Bigg\}\,, \tag{2.26}$$

One may prove that with appropriate boundary conditions, e.g. such those mentioned above so as boundary integrals to vanish, the identity

$$\int_D d^3x\,[f,g]h = \int_D d^3x\,[h,f]g = \int_D d^3x\,[g,h]f\,, \tag{2.27}$$

holds for arbitrary functions f, g, h. Making use of this identity we may write

$$\{X^1, X^2\}_{vort} = \epsilon_{jm}\int d^3x\ \times$$
$$\times\Bigg\{\rho^{-1}\left(k^{-2}\Omega - 2n\ell k v_h\right)\left(\frac{k^2}{2}[X^j_\Omega, X^m_\Omega] + \frac{1}{2}[X^j_w, X^m_w] + k^2\nabla X^j_w\cdot\nabla X^m_\Omega\right)$$
$$+\kappa^{-1}v_h\left([\rho^{-1}kX^j_{v_h}, X^m_\Omega] - \nabla(\rho^{-1}kX^j_{v_h})\cdot\nabla X^m_w - \rho^{-1}kX^j_{v_h}X^m_\Upsilon\right)\Bigg\}\,. \tag{2.28}$$

Now let us similarly analyze the reduction of the flow-magnetic field interaction part using (2.6), (2.17) and (2.22) to get after some analysis

$$\{X^1, X^2\}_{int} = \epsilon_{jm}\int d^3x\,\rho^{-1}\mathbf{B}^*\cdot\left[X^j_{\mathbf{v}}\times(\boldsymbol{\nabla}\times X^m_{\mathbf{B}^*})\right]$$
$$= \epsilon_{jm}\int d^3x\,\rho^{-1}\Bigg\{\left(k^{-1}B^*_h\mathbf{h} + \nabla\psi^*\times\mathbf{h}\right)\cdot\Bigg[kX^j_{v_h}\nabla(k^{-1}X^m_{B^*_h})$$
$$-(X^m_{\psi^*} - 2n\ell k^3X^m_{B^*_h})\nabla X^j_\Omega + [X^j_\Omega, k^{-1}X^m_{B^*_h}]\mathbf{h} - k^{-2}X^m_{\psi^*}\nabla X^j_w\times\mathbf{h}$$
$$+2n\ell kX^m_{B^*_h}\nabla X^j_w\times\mathbf{h} + \nabla X^j_w\cdot\nabla(k^{-1}X^m_{B^*_h})\mathbf{h}\Bigg]\Bigg\}$$
$$= \epsilon_{jm}\int d^3x\,\rho^{-1}\Bigg\{kB^*_h[X^j_\Omega, k^{-1}X^m_{B^*_h}] + kB^*_h\nabla X^j_w\cdot\nabla(k^{-1}X^m_{B^*_h})$$
$$-kX^j_{v_h}[\psi^*, k^{-1}X^m_{B^*_h}] + \left(X^m_{\psi^*} - 2n\ell k^3X^m_{B^*_h}\right)\left([\psi^*, X^j_\Omega] - \nabla\psi^*\cdot\nabla X^j_w\right)\Bigg\}\,. \tag{2.29}$$

Integrating by parts and omitting surface integrals we find

$$\{X^1, X^2\}_{int} = \epsilon_{jm}\int d^3x\,\Bigg\{k\rho^{-1}B^*_h\left([X^j_\Omega, k^{-1}X^m_{B^*_h}] + \nabla(k^{-1}X^m_{B^*_h})\cdot\nabla X^j_w\right)$$
$$+\psi^*\Bigg([\rho^{-1}kX^j_{v_h}, k^{-1}X^m_{B^*_h}] + [X^j_\Omega, \rho^{-1}X^m_{\psi^*}] + \nabla\cdot(\rho^{-1}X^m_{\psi*}\nabla X^j_w)$$
$$-2n\ell[X^j_\Omega, \rho^{-1}k^3X^m_{B^*_h}] - 2n\ell\nabla\cdot(\rho^{-1}k^3X^m_{B^*_h}\nabla X^j_w)\Bigg)\Bigg\}\,. \tag{2.30}$$

Proceeding with the Hall part, Eqs. (2.6) and (2.22) are invoked to obtain

$$\{X^1, X^2\}_{hall} = -d_i\frac{\epsilon_{jm}}{2}\int d^3x\rho^{-1}\mathbf{B}^*\cdot\left[(\nabla\times X^j_{\mathbf{B}^*})\times(\nabla\times X^m_{\mathbf{B}^*})\right]$$

$$= -d_i\epsilon_{jm}\int d^3x\rho^{-1}\left(k^{-1}B_h^*\mathbf{h}+\nabla\psi^*\times\mathbf{h}\right)\cdot\Big\{\frac{1}{2}[k^{-1}X^j_{B_h^*},k^{-1}X^m_{B_h^*}]\mathbf{h}$$
$$+\left(X^j_{\psi^*}-2n\ell k^3X^j_{B_h^*}\right)\nabla\left(k^{-1}X^m_{B_h^*}\right)\Big\}$$
$$= -d_i\epsilon_{jm}\int d^3x\Big\{\frac{kB_h^*}{2\rho}[k^{-1}X^j_{B_h^*},k^{-1}X^m_{B_h^*}]+\psi^*\Big([\rho^{-1}X^j_{\psi^*},k^{-1}X^m_{B_h^*}]$$
$$-2n\ell[\rho^{-1}k^3X^j_{B_h^*},k^{-1}X^m_{B_h^*}]\Big)\Big\}. \tag{2.31}$$

Lastly, we compute the electron inertial part

$$\{X^1,X^2\}_{inertial}=d_e^2\frac{\epsilon_{jm}}{2}\int d^3x\,\rho^{-1}(\nabla\times\mathbf{v})\cdot\Big[(\nabla\times X^j_{\mathbf{B}^*})\times(\nabla\times X^m_{\mathbf{B}^*})\Big]$$
$$= d_e^2\epsilon_{jm}\int d^3x\,\rho^{-1}\left[\left(k^{-2}\Omega-2n\ell kv_h\right)\mathbf{h}+\nabla(k^{-1}v_h)\times\mathbf{h}\right]\cdot$$
$$\cdot\Big\{\frac{1}{2}[k^{-1}X^j_{B_h^*},k^{-1}X^m_{B_h^*}]\mathbf{h}+\left(X^j_{\psi^*}-2n\ell k^3X^j_{B_h^*}\right)\nabla\left(k^{-1}X^m_{B_h^*}\right)\Big\}$$
$$= d_e^2\epsilon_{jm}\int d^3x\Big\{\frac{k}{2\rho}\left(k^{-2}\Omega-2n\ell kv_h\right)[k^{-1}X^j_{B_h^*},k^{-1}X^m_{B_h^*}]$$
$$+k^{-1}v_h\left([\rho^{-1}X^j_{\psi^*},k^{-1}X^m_{B_h^*}]-2n\ell[\rho^{-1}k^3X^j_{B_h^*},k^{-1}X^m_{B_h^*}]\right)\Big\}. \tag{2.32}$$

Putting all these terms together, and expanding the contracted summations, writing the result in terms of $F=X^1$ and $G=X^2$, we take the complete expression for the helically symmetric XMHD Poisson bracket

$$\begin{aligned}
\{F,G\}_{hs} \quad &= \int_D d^3x\Big\{F_\rho\Delta G_w-G_\rho\Delta F_w+\rho^{-1}\left(\Omega-2n\ell k^3v_h\right)\times\\
&\times\big([F_\Omega,G_\Omega]+k^{-2}[F_w,G_w]+\nabla F_w\cdot\nabla G_\Omega-\nabla F_\Omega\cdot\nabla G_w\big)\\
&+k^{-1}v_h\big([F_\Omega,\rho^{-1}kG_{v_h}]-[G_\Omega,\rho^{-1}kF_{v_h}]\\
&+\nabla\cdot(\rho^{-1}kG_{v_h}\nabla F_w)-\nabla\cdot(\rho^{-1}kF_{v_h}\nabla G_w)\big)\\
&+\rho^{-1}kB_h^*\big([F_\Omega,k^{-1}G_{B_h^*}]-[G_\Omega,k^{-1}F_{B_h^*}]\\
&+\nabla F_w\cdot\nabla\left(k^{-1}G_{B_h^*}\right)-\nabla G_w\cdot\nabla\left(k^{-1}F_{B_h^*}\right)\big)\\
&+\psi^*\big([F_\Omega,\rho^{-1}G_{\psi^*}]-[G_\Omega,\rho^{-1}F_{\psi^*}]+[k^{-1}F_{B_h^*},\rho^{-1}kG_{v_h}]\\
&-[k^{-1}G_{B_h^*},\rho^{-1}kF_{v_h}]+\nabla\cdot\left(\rho^{-1}G_{\psi^*}\nabla F_w\right)-\nabla\cdot\left(\rho^{-1}F_{\psi^*}\nabla G_w\right)\big)\\
&-2n\ell\psi^*\big([F_\Omega,\rho^{-1}k^3G_{B_h^*}]-[G_\Omega,\rho^{-1}k^3F_{B_h^*}]\\
&+\nabla\left(\rho^{-1}k^3G_{B_h^*}\nabla F_w\right)-\nabla\left(\rho^{-1}k^3F_{B_h^*}\nabla G_w\right)\big)\\
&-d_i\rho^{-1}kB_h^*[k^{-1}F_{B_h^*},k^{-1}G_{B_h^*}]\\
&-d_i\psi^*\big([\rho^{-1}F_{\psi^*},k^{-1}G_{B_h^*}]-[\rho^{-1}G_{\psi^*},k^{-1}F_{B_h^*}]\big)\\
&+2n\ell d_i\psi^*\big([\rho^{-1}k^3F_{B_h^*},k^{-1}G_{B_h^*}]-[\rho^{-1}k^3G_{B_h^*},k^{-1}F_{B_h^*}]\big)\\
&+d_e^2\rho^{-1}\left(\Omega-2n\ell k^3v_h\right)[k^{-1}F_{B_h^*},k^{-1}G_{B_h^*}]
\end{aligned}$$

$$+d_e^2 k^{-1} v_h \big([\rho^{-1} F_{\psi^*}, k^{-1} G_{B_h^*}] - [\rho^{-1} G_{\psi^*}, k^{-1} F_{B_h^*}]\big)$$
$$-2n\ell d_e^2 k^{-1} v_h \big([\rho^{-1} k^3 F_{B_h^*}, k^{-1} G_{B_h^*}] - [\rho^{-1} k^3 G_{B_h^*}, k^{-1} F_{B_h^*}]\big)\Big\} . \quad (2.33)$$

It's not difficult to show that if we set $a = \pi/2$ the bracket (2.33) reduces to the translationally symmetric XMHD bracket derived in [71]. The corresponding axisymmetric bracket can be obtained upon setting $a = 0$. In this case the purely helical terms, which contain a coefficient $2n\ell$ vanish and the scale factor k becomes $1/r$.

To complete the Hamiltonian description of helically symmetric XMHD dynamics we need to express the Hamiltonian (1.71) in terms of the scalar fields u_{HS}, leading to

$$\mathcal{H} = \int_D d^3x \left\{ \frac{\rho}{2} \left(v_h^2 + k^2 |\nabla \chi|^2 + |\nabla \Upsilon|^2 \right) \right.$$
$$\left. + \rho \left([\Upsilon, \chi] + U(\rho) \right) + \frac{B_h^* B_h}{2} + k^2 \frac{\nabla \psi^* \cdot \nabla \psi}{2} \right\} . \quad (2.34)$$

Also, from the definition of the generalized magnetic field $\mathbf{B}^*$ (1.30) and the helical representation (2.6) one can derive the following relations for the generalized variables B_h^* and ψ^*:

$$B_h^* = (1 + 4n^2\ell^2 d_e^2 \rho^{-1} k^4) B_h + d_e^2 \big[\rho^{-1} k^{-1} \mathcal{L}(k^{-1} B_h)$$
$$- 2n\ell \rho^{-1} k \mathcal{L}\psi - k \nabla \rho^{-1} \cdot \nabla (k^{-1} B_h) \big] , \quad (2.35)$$
$$\psi^* = \psi + d_e^2 \left[\rho^{-1} k^{-2} \mathcal{L}\psi - 2n\ell \rho^{-1} k B_h \right] , \quad (2.36)$$

where B_h is the helical component and ψ the poloidal flux function of the magnetic field $\mathbf{B}$. Note that terms containing the product $n\ell$ are purely helical, i.e., they vanish in the cases of axial and translational symmetry. Also, the last term of (2.35) is purely compressible, i.e., it vanishes if we consider incompressible plasmas. Another interesting observation is that due to the non-orthogonality of the helical coordinates, there is a poloidal magnetic field contribution in the helical component of the generalized magnetic field B_h^* and helical magnetic contribution B_h in the poloidal flux function ψ^*. This mixing makes the subsequent dynamical and equilibrium analyses appear much more involved than in [71], however it can be simplified upon observing that

$$\int_D d^3x \, [B_h^* \delta B_h + \mathcal{L}(\psi^*) \delta\psi]$$
$$= \int_D d^3x \left[B_h \delta B_h^* + \mathcal{L}(\psi) \delta\psi^* + \frac{d_e^2}{\rho^2} \left(J_h^2 + k^2 |\nabla (k^{-1} B_h)|^2 \right) \delta\rho \right] , \quad (2.37)$$

where $J_h = k^{-1} \mathcal{L}\psi - 2n\ell k^2 B_h$ is the helical component of the current density. Relation (2.37) can be easily found from the following procedure

$$\delta \int d^3x \, \mathbf{B}^* \cdot \mathbf{B} = \int d^3x \, (\mathbf{B}^* \cdot \delta\mathbf{B} + \delta\mathbf{B}^* \cdot \mathbf{B})$$

$$= \int d^3x \left(2\mathbf{B}^* \cdot \delta\mathbf{B} - \frac{d_e^2}{\rho^2}|\mathbf{J}|^2\delta\rho \right), \tag{2.38}$$

where the definition of $\mathbf{B}^*$ (1.30) has been used. From the second equality one has

$$\int d^3x\, \mathbf{B}^* \cdot \delta\mathbf{B} = \int d^3x \left(\mathbf{B} \cdot \delta\mathbf{B}^* + \frac{d_e^2}{\rho^2}|\mathbf{J}|^2\delta\rho \right). \tag{2.39}$$

Therefore, the variation of the magnetic part of the Hamiltonian can be written as

$$\begin{aligned} \delta\mathcal{H}_m &= \int_D d^3x \left[\frac{1}{2}B_h^*\delta B_h + \frac{1}{2}B_h\delta B_h^* + \frac{1}{2}\mathcal{L}(\psi^*)\delta\psi + \frac{1}{2}\mathcal{L}(\psi)\delta\psi^* \right] \\ &= \int_D d^3x \left[B_h\delta B_h^* + \mathcal{L}(\psi)\delta\psi^* + \frac{d_e^2}{2\rho^2}\left(J_h^2 + k^2|\nabla(k^{-1}B_h)|^2\right)\delta\rho \right] \\ &= \int_D d^3x \left[B_h^*\delta B_h + \mathcal{L}(\psi^*)\delta\psi - \frac{d_e^2}{2\rho^2}\left(J_h^2 + k^2|\nabla(k^{-1}B_h)|^2\right)\delta\rho \right], \end{aligned} \tag{2.40}$$

leading to the following relations for the functional derivatives of the Hamiltonian:

$$\frac{\delta\mathcal{H}}{\delta B_h} = B_h^*, \quad \frac{\delta\mathcal{H}}{\delta\psi} = \mathcal{L}\psi^*, \tag{2.41}$$

$$\frac{\delta\mathcal{H}}{\delta B_h^*} = B_h, \quad \frac{\delta\mathcal{H}}{\delta\psi^*} = \mathcal{L}\psi, \tag{2.42}$$

$$\left.\frac{\delta\mathcal{H}}{\delta\rho}\right|_{B_h^*,\psi^*} = \frac{|\mathbf{v}|^2}{2} + [\rho U(\rho)]_\rho + \frac{d_e^2}{2\rho^2}\left(J_h^2 + k^2|\nabla(k^{-1}B_h)|^2\right), \tag{2.43}$$

$$\left.\frac{\delta\mathcal{H}}{\delta\rho}\right|_{B_h,\psi} = \frac{|\mathbf{v}|^2}{2} + [\rho U(\rho)]_\rho - \frac{d_e^2}{2\rho^2}\left(J_h^2 + k^2|\nabla(k^{-1}B_h)|^2\right). \tag{2.44}$$

In addition, the functional derivatives with respect to the velocity related variables are given by

$$\frac{\delta\mathcal{H}}{\delta v_h} = \rho v_h, \quad \frac{\delta\mathcal{H}}{\delta\chi} = -\nabla\cdot(\rho k^2\nabla\chi) + [\rho, \Upsilon], \tag{2.45}$$

$$\frac{\delta\mathcal{H}}{\delta\Upsilon} = -\nabla\cdot(\rho\nabla\Upsilon) + [\chi, \rho], \quad \frac{\delta\mathcal{H}}{\delta\Omega} = \mathcal{L}^{-1}\frac{\delta\mathcal{H}}{\delta\chi}, \quad \frac{\delta\mathcal{H}}{\delta w} = \Delta^{-1}\frac{\delta\mathcal{H}}{\delta\Upsilon}. \tag{2.46}$$

2.1.1 Helically symmetric dynamics

The helically symmetric dynamics is described by means of the Hamiltonian (2.34) and the Poisson bracket (2.33) via $\partial_t\mathbf{u}_{HS} = \{\mathbf{u}_{HS}, \mathcal{H}\}_{HS}^{XMHD}$. Due to helical symmetry and compressibility, the equations of motion appear much more involved than the corresponding equations of motion in [53]. In view of (2.33) and (2.34) we have:

$$\begin{aligned} \partial_t\rho &= -\nabla\cdot(\rho\nabla\Upsilon) + [\chi, \rho], & (2.47) \\ \partial_t v_h &= \rho^{-1}k\left([\mathcal{H}_\Omega, k^{-1}v_h] + [k^{-1}B_h, \psi^*] + \nabla(k^{-1}v_h)\cdot\nabla\mathcal{H}_w\right), & (2.48) \\ \partial_t\Omega &= [\mathcal{H}_\Omega, \rho^{-1}\Omega] - 2n\ell[\mathcal{H}_\Omega, \rho^{-1}k^3v_h] + \nabla\cdot(\rho^{-1}\Omega\nabla\mathcal{H}_w) \end{aligned}$$

$$
\begin{aligned}
& -2n\ell\nabla\cdot(\rho^{-1}k^3 v_h \nabla\mathcal{H}_w) + [k v_h, k^{-1}v_h] + [k^{-1}B_h, \rho^{-1}kB_h^*] \\
& +[\rho^{-1}\mathcal{L}\psi, \psi^*] - 2n\ell[\rho^{-1}k^3 B_h, \psi^*]\,, \qquad (2.49)\\
\partial_t w \;=\; & -\Delta\mathcal{H}_\rho + [\mathcal{H}_w, \rho^{-1}k^{-2}\Omega] - 2n\ell[\mathcal{H}_w, \rho^{-1}k v_h] - \nabla\cdot(\rho^{-1}\Omega\nabla\mathcal{H}_\Omega) \\
& +2n\ell\nabla\cdot(\rho^{-1}k^3 v_h\nabla\mathcal{H}_\Omega) + \nabla\cdot\big(k v_h\nabla(k^{-1}v_h)\big) - \nabla\cdot(\rho^{-1}kB_h^*\nabla(k^{-1}B_h)) \\
& +\nabla\cdot(\rho^{-1}\mathcal{L}\psi\nabla\psi^*) - 2n\ell\nabla\cdot(\rho^{-1}k^3 B_h\nabla\psi^*)\,, \qquad (2.50)\\
\partial_t B_h^* \;=\; & k^{-1}\big([\mathcal{H}_\Omega, \rho^{-1}kB_h^*] + \nabla\cdot(\rho^{-1}kB_h^*\nabla\mathcal{H}_w) + [k v_h, \psi^*] - 2n\ell\rho^{-1}k^4[\mathcal{H}_\Omega, \psi^*] \\
& -2n\ell\rho^{-1}k^4\nabla\psi^*\cdot\nabla\mathcal{H}_w + d_i[\rho^{-1}kB_h^*, k^{-1}B_h] - d_i[\rho^{-1}\mathcal{L}\psi, \psi^*] \\
& -2n\ell d_i\rho^{-1}k^4[\psi^*, k^{-1}B_h] + 2n\ell d_i[\rho^{-1}k^3 B_h, \psi^*] + d_e^2[k^{-1}B_h, \rho^{-1}\Omega] \\
& -2n\ell d_e^2[k^{-1}B_h, \rho^{-1}k^3 v_h] + d_e^2[\rho^{-1}\mathcal{L}\psi, k^{-1}v_h] \\
& -2n\ell d_e^2\rho^{-1}k^4[k^{-1}B_h, k^{-1}v_h] - 2n\ell d_e^2[\rho^{-1}k^3 B_h, k^{-1}v_h]\big)\,, \qquad (2.51)\\
\partial_t\psi^* \;=\; & \rho^{-1}\left([\mathcal{H}_\Omega, \psi^*] + \nabla\psi^*\cdot\nabla\mathcal{H}_w + d_i[\psi^*, k^{-1}B_h] + d_e^2[k^{-1}B_h, k^{-1}v_h]\right)\,, \qquad (2.52)
\end{aligned}
$$

where $\mathcal{H}_\rho$ is given by (2.43) while $\mathcal{H}_\Omega$ and $\mathcal{H}_w$ are given by (2.46). Incompressible equations of motion are obtained from the corresponding Hamiltonian and Poisson bracket with $\rho = 1$ and $w = 0$, or equivalently by equations (2.47)–(2.52), upon neglecting the dynamical equations for ρ and w and substituting $\mathcal{H}_w = 0$ and $\mathcal{H}_\Omega = \chi$ in the rest, leading to the following system

$$
\begin{aligned}
\partial_t v_h \;& = k\left([\chi, k^{-1}v_h] + [k^{-1}B_h, \psi^*]\right)\,, \qquad (2.53)\\
\partial_t\Omega \;& = [\chi, \Omega] - 2n\ell[\chi, k^3 v_h] + [k v_h, k^{-1}v_h] \\
& +[k^{-1}B_h, kB_h^*] + [\mathcal{L}\psi, \psi^*] - 2n\ell[k^3 B_h, \psi^*]\,, \qquad (2.54)\\
\partial_t B_h^* \;& = k^{-1}\big([\chi, kB_h^*] + [k v_h, \psi^*] - 2n\ell k^4[\chi, \psi^*] + d_i[kB_h^*, k^{-1}B_h] \\
& -d_i[\mathcal{L}\psi, \psi^*] - 2n\ell d_i k^4[\psi^*, k^{-1}B_h] + 2n\ell d_i[k^3 B_h, \psi^*] \\
& +d_e^2[k^{-1}B_h, \Omega] - 2n\ell d_e^2[k^{-1}B_h, k^3 v_h] + d_e^2[\mathcal{L}\psi, k^{-1}v_h] \\
& -2n\ell d_e^2 k^4[k^{-1}B_h, k^{-1}v_h] - 2n\ell d_e^2[k^3 B_h, k^{-1}v_h]\big)\,, \qquad (2.55)\\
\partial_t\psi^* \;& = [\chi, \psi^*] + d_i[\psi^*, k^{-1}B_h] + d_e^2[k^{-1}B_h, k^{-1}v_h]\,. \qquad (2.56)
\end{aligned}
$$

Equations (2.53)–(2.56) differ from the corresponding dynamical equations of reference [53] owing to the presence of the scale factor k and the purely helical terms with the coefficients $n\ell$. Setting $n = 0$ we recover the equations of motion for incompressible, translationally symmetric plasmas, whereas for $\ell = 0$ we restrict the motion to respect axial symmetry.

2.1.2 Bracket transformation

As mentioned in Chapter 1, in [21] the authors proved that the XMHD bracket (1.72) can be simplified to a form identical to the HMHD bracket by introducing a generalized vorticity variable

$$\mathbf{B}^{\pm} = \mathbf{B}^{*} + \gamma_{\pm}\boldsymbol{\nabla} \times \mathbf{v}\,, \tag{2.57}$$

inducing transformation (1.76). This transformation was utilized in [53, 71] in order to simplify the bracket and consequently the derivation of the corresponding symmetric families of Casimir invariants. For this reason, we perform this transformation also for the helically symmetric bracket (2.33), rendering the subsequent analysis more tractable. One can see that the corresponding scalar field variables, necessary for the poloidal representation of $\mathbf{B}^{\pm}$, are connected to variables $\boldsymbol{u}_{HS}$ as follows:

$$B_h^{\pm} = B_h^{*} + \gamma_{\pm}(k^{-1}\Omega - 2n\ell k^2 v_h)\,, \tag{2.58}$$

$$\psi^{\pm} = \psi^{*} + \gamma_{\pm}k^{-1}v_h\,. \tag{2.59}$$

Transforming the bracket requires the expressions of functional derivatives in the new representation $(v_h, \chi, \Upsilon, B_h^{\pm}, \psi^{\pm})$. Following an analogous procedure to that employed in [21, 53, 71] we find

$$\bar{F}_{v_h} = F_{v_h} + \gamma_{\pm}k^{-1}F_{\psi^{\pm}} - 2n\ell\gamma_{\pm}k^2 F_{B_h^{\pm}}\,, \tag{2.60}$$

$$\bar{F}_{\Omega} = F_{\Omega} + \gamma_{\pm}k^{-1}F_{B_h^{\pm}}\,, \quad \bar{F}_w = F_w\,, \tag{2.61}$$

$$\bar{F}_{\psi^*} = F_{\psi^{\pm}}\,, \quad \bar{F}_{B_h^*} = F_{B_h^{\pm}}\,, \tag{2.62}$$

where $\bar{F}$ denotes the functionals depending on the original variables. Upon inserting the transformation of the functional derivatives given by (2.60)–(2.62) into (2.33) and expressing B_h^* and ψ^* in terms of $B_h^{\pm}$ and $\psi^{\pm}$, we obtain the following bracket:

$$\begin{aligned}
\{F,G\}_{HS}^{XMHD} = \int_D d^3x \Big\{ & F_\rho \Delta G_w - G_\rho \Delta F_w + \rho^{-1}(\Omega - 2n\ell k^3 v_h) \times \\
& \times\big([F_\Omega, G_\Omega] + k^{-2}[F_w, G_w] + \nabla F_w \cdot \nabla G_\Omega - \nabla F_\Omega \cdot \nabla G_w\big) \\
& + k^{-1}v_h\big([\rho^{-1}kF_{v_h}, G_\Omega] - [\rho^{-1}kG_{v_h}, F_\Omega] \\
& + \nabla\cdot(\rho^{-1}kG_{v_h}\nabla F_w) - \nabla\cdot(\rho^{-1}kF_{v_h}\nabla G_w)\big) \\
& + \rho^{-1}kB_h^{\pm}\Big([F_\Omega, k^{-1}G_{B_h^{\pm}}] - [G_\Omega, k^{-1}F_{B_h^{\pm}}] \\
& + \nabla F_w \cdot \nabla(k^{-1}G_{B_h^{\pm}}) - \nabla G_w \cdot \nabla(k^{-1}F_{B_h^{\pm}})\Big) \\
& + \psi^{\pm}\big([F_\Omega, \rho^{-1}G_{\psi^{\pm}}] - [G_\Omega, \rho^{-1}F_{\psi^{\pm}}] \\
& + [\rho^{-1}kF_{v_h}, k^{-1}G_{B_h^{\pm}}] - [\rho^{-1}kG_{v_h}, k^{-1}F_{B_h^{\pm}}] \\
& + \nabla\cdot(\rho^{-1}G_{\psi^{\pm}}\nabla F_w) - \nabla\cdot(\rho^{-1}F_{\psi^{\pm}}\nabla G_w)\big) \\
& - 2n\ell\psi^{\pm}\big([F_\Omega, \rho^{-1}k^3G_{B_h^{\pm}}] - [G_\Omega, \rho^{-1}k^3F_{B_h^{\pm}}] \\
& + \nabla\cdot(\rho^{-1}k^3G_{B_h^{\pm}}\nabla F_w) - \nabla\cdot(\rho^{-1}k^3F_{B_h^{\pm}}\nabla G_w)\big) \\
& - \nu_{\pm}\rho^{-1}kB_h^{\pm}[k^{-1}F_{B_h^{\pm}}, k^{-1}G_{B_h^{\pm}}]
\end{aligned}$$

$$-\nu_\pm\psi^\pm\Big([\rho^{-1}F_{\psi^\pm},k^{-1}G_{B_h^\pm}]-[\rho^{-1}G_{\psi^\pm},k^{-1}F_{B_h^\pm}]\Big)$$
$$+2n\ell\nu_\pm\psi^\pm\Big([\rho^{-1}k^3F_{B_h^\pm},k^{-1}G_{B_h^\pm}]-[\rho^{-1}k^3G_{B_h^\pm},k^{-1}F_{B_h^\pm}]\Big)\Big\}\,, \quad (2.63)$$

where $\nu_\pm := d_i - 2\gamma_\pm$. Note that the helically symmetric XMHD dynamics is described correctly by either using the parameter ν_+ or the parameter ν_-.

2.2 Helically symmetric Casimir invariants

2.2.1 Casimir determining equations

As mentioned in the Introduction, the Casimir invariants are functionals that satisfy $\{F,\mathcal{C}\}=0$, $\forall F$. For bracket (2.63) this condition is equivalent to

$$\int_D d^3x\,\big(F_\rho\mathfrak{R}_1+F_w\mathfrak{R}_2+\rho^{-1}kF_{v_h}\mathfrak{R}_3+F_\Omega\mathfrak{R}_4+k^{-1}F_{B_h^\pm}\mathfrak{R}_5+\rho^{-1}F_{\psi^\pm}\mathfrak{R}_6\big)=0\,, \quad (2.64)$$

where $\mathfrak{R}_i\,, i=1,...,6$ are expressions obtained by manipulating the bracket $\{F,\mathcal{C}\}$ so as to extract as common factors the functional derivatives of the arbitrary functional F. Requiring (2.64) to be satisfied for arbitrary variations is equivalent to the independent vanishing of the expressions $\mathfrak{R}_i$, i.e.,

$$\mathfrak{R}_i=0\,,\qquad i=1,2,...,6\,. \quad (2.65)$$

The expressions for $\mathfrak{R}_i\,, i=1,...6$, read as

$$\mathfrak{R}_1 = \Delta\mathcal{C}_w=\mathcal{C}_\Upsilon\,, \quad (2.66)$$
$$\begin{aligned}\mathfrak{R}_2 =& -\Delta\mathcal{C}_\rho-[\rho^{-1}k^{-2}\Omega,\mathcal{C}_w]+2n\ell[\rho^{-1}kv_h,\mathcal{C}_w]\\ &+\nabla\cdot(\rho^{-1}\mathcal{C}_{\psi^\pm}\nabla\psi^\pm)-2n\ell\nabla\cdot(\rho^{-1}k^3\mathcal{C}_{B_h^\pm}\nabla\psi^\pm)\\ &+\nabla\cdot(\rho^{-1}k\mathcal{C}_{v_h}\nabla(k^{-1}v_h))-\nabla\cdot(\rho^{-1}\Omega\nabla\mathcal{C}_\Omega)\\ &+2n\ell\nabla\cdot(\rho^{-1}k^3v_h\nabla\mathcal{C}_\Omega)-\nabla\cdot(\rho^{-1}kB_h^\pm\nabla(k^{-1}\mathcal{C}_{B_h^\pm}))\,,\end{aligned} \quad (2.67)$$
$$\mathfrak{R}_3 = [\mathcal{C}_\Omega,k^{-1}v_h]+\nabla(k^{-1}v_h)\cdot\nabla\mathcal{C}_w-[\psi_\pm,k^{-1}\mathcal{C}_{B_h^\pm}]\,, \quad (2.68)$$
$$\begin{aligned}\mathfrak{R}_4 =& \nabla\cdot(\rho^{-1}\Omega\nabla\mathcal{C}_w)-2n\ell\nabla\cdot(\rho^{-1}k^3v_h\nabla\mathcal{C}_w)\\ &-[\rho^{-1}\Omega,\mathcal{C}_\Omega]+2n\ell[\rho^{-1}k^3v_h,\mathcal{C}_\Omega]-[k^{-1}v_h,\rho^{-1}k\mathcal{C}_{v_h}]\\ &-[\psi^\pm,\rho^{-1}\mathcal{C}_{\psi^\pm}]-[\rho^{-1}kB_h^\pm,k^{-1}\mathcal{C}_{B_h^\pm}]+2n\ell[\psi^\pm,\rho^{-1}k^3\mathcal{C}_{B_h^\pm}]\,,\end{aligned} \quad (2.69)$$
$$\begin{aligned}\mathfrak{R}_5 =& [\rho^{-1}k\mathcal{C}_{v_h},\psi^\pm]+[\mathcal{C}_\Omega,\rho^{-1}kB_h^\pm]+\nabla\cdot(\rho^{-1}kB_h^\pm\nabla\mathcal{C}_w)-2n\ell\rho^{-1}k^4[\mathcal{C}_\Omega,\psi^\pm]\\ &-2n\ell\rho^{-1}k^4\nabla\psi^\pm\cdot\nabla\mathcal{C}_w+\nu_\pm[\psi_\pm,\rho^{-1}\mathcal{C}_{\psi_\pm}]+\nu_\pm[\rho^{-1}kB_h^\pm,k^{-1}\mathcal{C}_{B_h^\pm}]\\ &-2n\ell\nu_\pm\rho^{-1}k^4[\psi^\pm,k^{-1}\mathcal{C}_{B_h^\pm}]+2n\ell\nu_\pm[\rho^{-1}k^3\mathcal{C}_{B_h^\pm},\psi^\pm]\,,\end{aligned} \quad (2.70)$$
$$\mathfrak{R}_6 = [\mathcal{C}_\Omega,\psi_\pm]+\nabla\psi_\pm\cdot\nabla\mathcal{C}_w+\nu_\pm[\psi_\pm,k^{-1}\mathcal{C}_{B_h^\pm}]\,. \quad (2.71)$$

Equation $\mathfrak{R}_1 = 0$, i.e. $\mathcal{C}_\Upsilon = 0$, implies that the Casimirs are independent of Υ, or equivalently $\mathcal{C}_w$ is equal either to a constant or to a harmonic function. In the former case the resulting Casimir is $\mathcal{C} = \int d^3x \Delta\Upsilon = \int d^3x \nabla\cdot\mathbf{v} = \oint \mathbf{v}\cdot\mathbf{dS} = 0$, hence it does not contribute to the subsequent variational principles. In the latter case, where a harmonic function comes into play, the remaining Casimir determining equations will introduce constraints between the field variables and this harmonic function. Hence, we conclude that the most general case corresponds to $\mathcal{C}_w = 0$. In addition, we observe that (2.65) are satisfied automatically for $\mathcal{C}_\rho$=const., which amounts to the conservation of mass density,

$$\mathcal{C}_m = \int_D d^3x\,\rho\,. \tag{2.72}$$

For the rest of the Casimirs we follow a similar procedure as in Section IIIB of [71].

To set a starting point for our derivation we observe that we can obtain simplified determining equations by taking linear combinations of $\mathfrak{R}_3$ with $\mathfrak{R}_6$, and $\mathfrak{R}_4$ with $\mathfrak{R}_5$. More specifically we have

$$\mathfrak{R}_6 + \nu_\pm \mathfrak{R}_3 = \rho^{-1}[\mathcal{C}_\Omega, \psi^\pm + \nu_\pm k^{-1} v_h] = 0\,, \tag{2.73}$$

which immediately indicates solutions of the form

$$\mathcal{C}_\pm = \int d^3x\,\Omega\mathcal{Z}_\pm(\psi^\pm + \nu_\pm k^{-1}v_h) + \int d^3x\,\mathcal{K}_\pm(\rho, v_h, \psi^\pm, B_h^\pm]\,, \tag{2.74}$$

where $\mathcal{Z}_\pm$ and $\mathcal{K}_\pm$ are arbitrary functions. Substituting (2.74) to $\mathfrak{R}_6 = 0$ alone we have

$$[\mathcal{Z}_\pm - \nu_\pm k^{-1}\frac{\partial\mathcal{K}_\pm}{\partial B_h^\pm}, \psi^\pm] = 0\,, \tag{2.75}$$

with solution

$$\mathcal{K}_\pm = \frac{k}{\nu_\pm} B_h^\pm \left[\mathcal{Z}_\pm - f_\pm(\psi^\pm)\right] + g_\pm(\rho, v_h, \psi^\pm)\,, \tag{2.76}$$

where $f_\pm$, $g_\pm$ are new arbitrary functions. Therefore, up to now the solution of the Casimir determining equations is of the form

$$\begin{aligned}\mathcal{C}_\pm = \int d^3x \,\big[\, &\left(kB_h^\pm + \nu_\pm\Omega\right)\mathcal{Z}_\pm(\psi^\pm + \nu_\pm k^{-1}v_h) \\ &-kB_h^\pm f_\pm(\psi^\pm) + g_\pm(\rho, v_h, \psi^\pm)\big]\,,\end{aligned} \tag{2.77}$$

where $\mathcal{Z}_\pm$, $f_\pm$ have been rescaled. Now taking the linear combination $\mathfrak{R}_5 + \nu_\pm\mathfrak{R}_4$ we have

$$[\mathcal{C}_\Omega, \rho^{-1}(kB_h^\pm + \nu_\pm\Omega)] + [\rho^{-1}k\mathcal{C}_{v_h}, (\psi^\pm + \nu_\pm k^{-1}v_h)] + 2n\ell\nu_\pm[\rho^{-1}k^3 v_h, \mathcal{C}_\Omega]$$

$$-2n\ell\nu_{\pm}\rho^{-1}k^{4}[\psi^{\pm},k^{-1}\mathcal{C}_{B_h^{\pm}}]+2n\ell\rho^{-1}k^{4}[\psi^{\pm},\mathcal{C}_{\Omega}]=0\,. \quad (2.78)$$

Upon substituting (2.77), Eq. (2.78) yields

$$[\rho^{-1}k\frac{\partial g_{\pm}}{\partial v_h},(\psi^{\pm}+\nu_{\pm}k^{-1}v_h)]+2n\ell\nu_{\pm}^{2}[\rho^{-1}k^{3}v_h,\mathcal{Z}_{\pm}]=0\,, \quad (2.79)$$

which is solved by

$$g_{\pm}=-2n\ell\nu_{\pm}k^{3}v_h\mathcal{Z}_{\pm}+2nlk^{4}\int_{0}^{\psi^{\pm}+\nu_{\pm}k^{-1}v_h}\mathcal{Z}_{\pm}(s)ds\,. \quad (2.80)$$

Therefore, the solution up to this point takes the form

$$\mathcal{C}_{\pm}=\int d^{3}x\Big[\left(kB_h^{\pm}+\nu_{\pm}\Omega-2n\ell\nu_{\pm}k^{3}v_h\right)\mathcal{Z}_{\pm}(\psi^{\pm}+\nu_{\pm}k^{-1}v_h)$$
$$+2n\ell k^{4}\int_{0}^{\psi^{\pm}+\nu_{\pm}k^{-1}v_h}\mathcal{Z}_{\pm}(s)ds-kB_h^{\pm}f_{\pm}(\psi^{\pm})\Big]\,. \quad (2.81)$$

Considering either $\mathfrak{R}_4=0$ or $\mathfrak{R}_5=0$ with $\mathcal{C}_{\pm}$ given by (2.81) the following result occurs

$$f_{\pm}[\rho^{-1}k^{4},\psi^{\pm}]=0\,, \quad (2.82)$$

which holds either if $f_{\pm}=0$ or if $[\rho^{-1}k^{4},\psi^{\pm}]=0$. The latter condition though, inserts a restriction in the dynamics connecting the flux functions $\psi^{\pm}$ with the mass density, therefore the more general case is given by $f_{\pm}=0$. In view of this condition our solution takes the form

$$\mathcal{C}_{\pm}=\int d^{3}x\Big[\left(kB_h^{\pm}+\nu_{\pm}\Omega-2n\ell\nu_{\pm}v_h\right)\mathcal{Z}_{\pm}(\psi^{\pm}+\nu_{\pm}k^{-1}v_h)$$
$$+2n\ell k^{4}\int_{0}^{\psi^{\pm}+\nu_{\pm}k^{-1}v_h}\mathcal{Z}_{\pm}(s)ds\Big]\,. \quad (2.83)$$

One can easily corroborate that (2.83) satisfies also $\mathfrak{R}_2=0$, and thus (2.83) is a solution to the complete set of the Casimir determining equations, that is $\mathcal{C}_{\pm}$ are truly helically symmetric XMHD Casimir invariants, however, they are not the only ones. Recall that our derivation started from Eq. (2.73), which is trivially satisfied if $\mathcal{C}_{\Omega}=0$, i.e. this equation is a good starting point for finding Casimirs that depend on Ω, but cannot provide any information about Casimirs that are Ω-independent. To consider such a class of invariants let us assume that there exist solutions to the Casimir determining equations, that do not dependent on Ω. We have also to assume that they are $B_h^{\pm}$-independent because the $B_h^{\pm}$-dependent case is included in the previous calculation. Thus, we consider a class of Casimirs which depend on ρ, v_h and $\psi^{\pm}$.

Equations $\mathfrak{R}_3 = 0$, $\mathfrak{R}_6 = 0$ are trivially satisfied, while from Eq. (2.78) we take

$$[\rho^{-1}k\mathcal{C}_{v_h}, (\psi^{\pm} + \nu_{\pm}k^{-1}v_h)] = 0\,, \tag{2.84}$$

with solution

$$\mathcal{C}_{\pm} = \int d^3x\, \rho\mathcal{X}_{\pm}(\psi^{\pm} + \nu_{\pm}k^{-1}v_h)\,, \tag{2.85}$$

where $\mathcal{X}_{\pm}$ are arbitrary functions. One can easily see that $\mathfrak{R}_4 = 0$, $\mathfrak{R}_5 = 0$ and $\mathfrak{R}_2 = 0$ are satisfied as well, which means that (2.85) represents two additional families of helically symmetric XMHD Casimirs.

Therefore, using Eqs. (2.58)–(2.59) and $\nu_{\pm} = d_i - 2\gamma_{\pm}$, the complete set of Casimirs in terms of the original generalized magnetic field variables (B_h^*, ψ^*) can be written as

$$\mathcal{C}_1 = \int_D d^3x \Big[(kB_h^* + \gamma\Omega - 2n\ell\gamma k^3 v_h)\mathcal{F}(\psi^* + \gamma k^{-1}v_h) + 2n\ell k^4 \int_0^{\psi^*+\gamma k^{-1}v_h} \mathcal{F}(s)ds\Big]\,, \tag{2.86}$$

$$\mathcal{C}_2 = \int_D d^3x \Big[(kB_h^* + \mu\Omega - 2n\ell\mu k^3 v_h)\mathcal{G}(\psi^* + \mu k^{-1}v_h) + 2n\ell k^4 \int_0^{\psi^*+\mu k^{-1}v_h} \mathcal{G}(s)ds\Big]\,, \tag{2.87}$$

$$\mathcal{C}_3 = \int_D d^3x\, \rho\mathcal{M}(\psi^* + \gamma k^{-1}v_h)\,, \tag{2.88}$$

$$\mathcal{C}_4 = \int_D d^3x\, \rho\mathcal{N}(\psi^* + \mu k^{-1}v_h)\,, \tag{2.89}$$

where the parameters γ and μ are $(\gamma,\mu) = (\gamma_+, \gamma_-)$, $\mathcal{F} = \mathcal{Z}_-$, $\mathcal{G} = \mathcal{Z}_+$, $\mathcal{M} = \mathcal{X}_-$, $\mathcal{N} = \mathcal{X}_+$ are arbitrary functions. We introduce this new notation because in the subsequent analysis, when taking the HMHD and MHD limits, the origin of $\pm$ subscripts will seem inexplicable.

Obviously $\mathcal{C}_m$ is just a special case of the functionals $\mathcal{C}_3$, $\mathcal{C}_4$. The interesting new feature of these Casimirs is the presence of two purely helical terms appearing in $\mathcal{C}_1$ and $\mathcal{C}_2$, which vanish for either $n = 0$ or $\ell = 0$. An analogous helical term, that depend on ψ, also having a coefficient $2n\ell$, appears in the Casimirs of ordinary MHD [70]. In the case of XMHD the helical terms depend on ψ^* and on the helical velocity v_h, this additional dependence on v_h emerges due to the presence of the vorticity in (1.74).

2.2.2 Inertial MHD limit

Inertial MHD (IMHD) arises in the limit $d_i \to 0$ with $d_e \neq 0$, which might be the case for processes with time scales shorter than the electron gyro-period and has applications in electron magnetic reconnection where very often spatially reduced models are considered. Hence, it is of interest to consider the IMHD limit of the Casimirs (2.86)–(2.89). For $d_i = 0$, $\gamma = d_e$ and $\mu = -d_e$, therefore the helically symmetric IMHD Casimirs are given by $\mathcal{C}_1^{IMHD} = \mathcal{C}_1|_{\gamma\to d_e}$, $\mathcal{C}_2^{IMHD} = \mathcal{C}_2|_{\mu\to -d_e}$, $\mathcal{C}_3^{IMHD} = \mathcal{C}_3|_{\gamma\to d_e}$ and $\mathcal{C}_4^{IMHD} = \mathcal{C}_4|_{\mu\to -d_e}$.

2.2.3 Hall MHD and MHD limits

Here, to validate once more that the computed invariants are correct, we take the MHD limit, anticipating the recovery of the invariants found in [70]. For the MHD limit we set $d_e = 0$ (Hall MHD) and then $d_i = 0$. Setting only $d_e = 0$ we exclude electron inertial contributions and we obtain the Hall MHD Casimirs

$$\mathcal{C}_1^{HMHD} = \int_D d^3x \Big[(kB_h + d_i\Omega - 2n\ell d_i k^3 v_h)\mathcal{F}(\psi + d_i k^{-1} v_h) + 2n\ell k^4 \tilde{\mathcal{F}}(\psi + d_i k^{-1} v_h)\Big] , \quad (2.90)$$

$$\mathcal{C}_2^{HMHD} = \int_D d^3x \Big[kB_h \mathcal{G}(\psi) + 2n\ell k^4 \tilde{\mathcal{G}}(\psi)\Big] , \quad (2.91)$$

$$\mathcal{C}_3^{HMHD} = \int_D d^3x \, \rho \mathcal{M}(\psi + d_i k^{-1} v_h) , \quad (2.92)$$

$$\mathcal{C}_4^{HMHD} = \int_D d^3x \, \rho \mathcal{N}(\psi) . \quad (2.93)$$

where $\tilde{\mathcal{F}}(\psi + d_i k^{-1} v_h) = \int_0^{\psi + d_i k^{-1} v_h} \mathcal{F}(s) ds$ and $\tilde{\mathcal{G}}(\psi) = \int_0^{\psi} \mathcal{G}(s) ds$. For the corresponding MHD families of invariants we additionally require $d_i \to 0$ in (2.90)–(2.93). From the resulting set of Casimirs, those related to the cross-helicity and the helical momentum are absent since in this limit $\mathcal{C}_1$ and $\mathcal{C}_3$ represent the same families of Casimir invariants with $\mathcal{C}_2$ and $\mathcal{C}_4$, respectively. This is a characteristic peculiarity, encountered when the MHD limit of models with Hall physics contributions is considered (e.g. see [71, 72, 18, 73]). This peculiarity is related to the fact that the Hall MHD equilibrium equations consist a singular perturbation problem [74, 75, 76], that requires special treatment when the MHD limit is considered. We can resolve this problem [71] by considering d_i as a small perturbation parameter and consequently performing perturbative expansions of the Casimirs $\mathcal{C}_1^{HMHD}$ and $\mathcal{C}_3^{HMHD}$

$$\mathcal{C}_1 = \int d^3x \left[(kB_h + d_i\Omega)\mathcal{F}(\psi) + d_i B_h v_h \mathcal{F}'(\psi) + 2n\ell k^4 \tilde{\mathcal{F}}(\psi) + \mathcal{O}(d_i^2)\right] , \quad (2.94)$$

$$\mathcal{C}_3 = \int d^3x \, \left[\rho\mathcal{M}(\psi) + d_i k^{-1} \rho v_h \mathcal{M}'(\psi) + \mathcal{O}(d_i^2)\right] . \quad (2.95)$$

Rescaling the arbitrary functions $\mathcal{F}$, $\mathcal{M}$ by a factor of d_i we get

$$\mathcal{C}_1 = \int d^3x \, d_i^{-1} \left[(kB_h + d_i\Omega)\mathcal{F}(\psi) + d_i B_h v_h \mathcal{F}'(\psi)\right.$$

$$+2n\ell k^4\tilde{\mathcal{F}}(\psi)+\mathcal{O}(d_i^2)\big]\,, \tag{2.96}$$

$$\mathcal{C}_3 = \int d^3x\, d_i^{-1}\left[\rho\mathcal{M}(\psi)+d_ik^{-1}\rho v_h\mathcal{M}'(\psi)+\mathcal{O}(d_i^2)\right]\,. \tag{2.97}$$

The terms that seem to diverge when $d_i \to 0$ are already Casimirs so they can be subtracted. The zeroth order terms translate to the absent MHD Casimirs while the higher order ones vanish in the limit $d_i \to 0$, leading to the following complete set of invariants

$$\mathcal{C}_1^{MHD} = \int_D d^3x\,\left[B_hv_h\mathcal{F}'(\psi)+\Omega\mathcal{F}(\psi)\right]\,, \tag{2.98}$$

$$\mathcal{C}_2^{MHD} = \int_D d^3x\,\left[kB_h\mathcal{G}(\psi)+2n\ell k^4\tilde{\mathcal{G}}(\psi)\right]\,, \tag{2.99}$$

$$\mathcal{C}_3^{MHD} = \int_D d^3x\,\rho k^{-1}v_h\mathcal{M}(\psi)\,, \tag{2.100}$$

$$\mathcal{C}_4^{MHD} = \int_D d^3x\,\rho\mathcal{N}(\psi)\,. \tag{2.101}$$

Functionals (2.98)–(2.101) are indeed the correct helically symmetric MHD Casimir invariants [70].

In the next chapter we exploit invariants (2.86)–(2.89) in order to derive equilibrium equations which describe XMHD equilibria with helical and axial symmetry employing the energy-Casimir variational principle (1.52). Analogous variational principles have been utilized by many authors over the last several decades for MHD and later on for Hall MHD. Usually the three dimensional counterparts of the Casimirs were considered. This is not the case however in more recent works e.g. [69] and [70] where the symmetric versions are used, leading to more general Grad-Shafranov equations. The merit of using the symmetric versions of the Casimirs instead of the spatially reduced three-dimensional versions, when dealing with symmetric problems, is that the former constitute infinite families of invariants due to the presence of arbitrary functions in their integral expressions. This allows for the construction of infinite sets of equilibria including those corresponding to the spatially reduced 3-D Casimirs, which are obtained by the symmetric ones upon choosing only linear free functions appearing in their integrands. The difference between the two settings can be easily understood upon comparing a simpler model, e.g. static axisymmetric MHD equilibria that is governed by the well known Grad-Shafranov equation [3]

$$\Delta^*\psi + r^2p'(\psi)+II'(\psi)=0\,, \tag{2.102}$$

where $I = rB_\phi$, $p(\psi)$ is the pressure function and $\Delta^* := r^2\nabla\cdot\left(\nabla/r^2\right)$ the Shafranov operator. If we try to determine the static MHD equilibrium equations by the 3D variational principle

$$\delta \int d^3x \, \frac{1}{2} \left(|\mathbf{B}|^2 - \lambda \mathbf{A} \cdot \mathbf{B} \right) = 0 \,, \tag{2.103}$$

with λ being a Lagrangian multiplier, we end up with a Beltrami magnetic field

$$\nabla \times \mathbf{B} = \lambda \mathbf{B} \,, \tag{2.104}$$

associated with the so-called Taylor's relaxed states [47]. Upon imposing axial symmetry this equation leads to

$$\Delta^* \psi + \lambda^2 \psi + \lambda I_0 = 0 \,, \tag{2.105}$$

where $I_0 = r B_0$, (B_0 is the vacuum field), which is different from (2.102), since it corresponds to $I = \lambda \psi + I_0$ and $p'(\psi) = 0$. If on the other hand we impose axial symmetry in the MHD Poisson bracket and dismiss the flow variables we will find two families of Casimir invariants of the form

$$\mathcal{C}_1 = \int d^3x \, r^{-1} B_\phi I(\psi) \,, \quad \mathcal{C}_2 = \int d^3x \, p(\psi) \,. \tag{2.106}$$

In this case the energy-Casimir variational principle will lead us to (2.102), i.e. it is capable of describing general classes of equilibria and not only linear ones corresponding to the relaxed state. This simple example justifies the utilization of symmetric formulations like those employed in the present thesis.

Chapter 3

Extended MHD Equilibria

In this chapter, results published in [68], [71] and [77] are presented as follows: In Section 3.1 the energy-Casimir variational principle (1.52), is applied for 3D and helically symmetric barotropic XMHD. In Section 3.2, we cast the resulting equilibrium equations in the form of a Grad-Shafranov-Bernoulli system and special cases are discussed. In addition, we present a numerical solution for the axisymmetric HMHD equilibrium equations. Section 3.3 deals with the incompressible case and an analytic double-Beltrami solution is constructed. Finally, in Section 3.4 the ellipticity condition for axisymmetric, barotropic XMHD equilibrium equations is derived.

3.1 The Energy-Casimir variational principle

3.1.1 Three-dimensional equilibrium and the triple-Beltrami states

Let us briefly recapitulate some of the basic ingredients which are introduced and discussed in Chapter 1. These are the EC equilibrium variational principle

$$\delta\left(\mathcal{H}-\sum_j \mathcal{C}_j\right)[\boldsymbol{u}_e]=0\,, \tag{3.1}$$

and the XMHD Hamiltonian and Casimir invariants, given by

$$\mathcal{H}=\int d^3x\,\left[\frac{1}{2}\rho|\mathbf{v}|^2+\rho U(\rho)+\frac{\mathbf{B}\cdot\mathbf{B}^*}{2}\right]\,, \tag{3.2}$$

and

$$\mathcal{C}=\frac{1}{2}\int d^3x\,(\mathbf{A}^*+\gamma_\pm\mathbf{v})\cdot(\mathbf{B}^*+\gamma_\pm\nabla\times\mathbf{v})\,,\quad \mathcal{C}_m=\int d^3x\,\rho\,, \tag{3.3}$$

respectively, where $\mathbf{A}^* = \mathbf{A} + d_e^2 \frac{\nabla\times\mathbf{B}}{\rho}$, $\mathbf{B}^* = \nabla \times \mathbf{A}^*$. From these ingredients, we can find in a variational manner, equilibrium equations governing general three-dimensional XMHD equilibria. The EC variational principle reads as

$$\begin{aligned}
&\delta \int d^3x \left[\frac{\rho}{2}|\mathbf{v}|^2 + \rho U(\rho) + \frac{\mathbf{B}^*\cdot\mathbf{B}}{2} - \sum_{\pm} \lambda_{\pm}(\mathbf{A}^* + \gamma_{\pm}\mathbf{v})\cdot(\mathbf{B}^* + \gamma_{\pm}\nabla\times\mathbf{v}) - \beta\rho\right] \\
&= \int d^3x \left\{ \left[\frac{|\mathbf{v}|^2}{2} + h(\rho) + \frac{d_e^2}{2}\frac{|\mathbf{J}|^2}{\rho^2} - \beta\right]\delta\rho + \rho\mathbf{v}\cdot\delta\mathbf{v} + (\nabla\times\mathbf{B})\cdot\delta\mathbf{A}^* \right. \\
&\left. - \sum_{\pm}\left[\lambda_{\pm}(\mathbf{B}^* + \gamma_{\pm}\nabla\times\mathbf{v})\cdot\delta\mathbf{A}^* + \lambda_{\pm}\gamma_{\pm}(\mathbf{B}^* + \gamma_{\pm}\nabla\times\mathbf{v})\cdot\delta\mathbf{v}\right] \right\} = 0\,,
\end{aligned} \tag{3.4}$$

where we have introduced the Lagrangian multipliers $\lambda_{\pm}$, β and the standard boundary conditions, $\delta\mathbf{A}^*|_{\partial V} = \delta\mathbf{v}|_{\partial V} = 0$, have been assumed. From (3.4) we obtain the following equilibrium equations

$$h(\rho) = \beta - \frac{|\mathbf{v}|^2}{2} - \frac{d_e^2}{2}\frac{|\mathbf{J}|^2}{\rho^2}\,, \tag{3.5}$$

$$\nabla\times\mathbf{B} = \lambda_+(\mathbf{B}^* + \gamma_+\nabla\times\mathbf{v}) + \lambda_-(\mathbf{B}^* + \gamma_-\nabla\times\mathbf{v})\,, \tag{3.6}$$

$$\rho\mathbf{v} = \lambda_+\gamma_+(\mathbf{B}^* + \gamma_+\nabla\times\mathbf{v}) + \lambda_-\gamma_-(\mathbf{B}^* + \gamma_-\nabla\times\mathbf{v})\,. \tag{3.7}$$

If we consider incompressible plasma i.e. $\rho = 1$ and $\delta\rho = 0$, then the variational principle (3.4) leads to

$$\nabla\times\mathbf{B} = \alpha_1\mathbf{B} + d_e^2\alpha_1\nabla\times\nabla\times\mathbf{B} + \alpha_2\nabla\times\mathbf{v}\,, \tag{3.8}$$

$$\mathbf{v} = \alpha_2\mathbf{B} + d_e^2\alpha_2\nabla\times\nabla\times\mathbf{B} + \alpha_3\nabla\times\mathbf{v}\,, \tag{3.9}$$

$$\alpha_1 = \lambda_+ + \lambda_-\,, \quad \alpha_2 = \lambda_+\gamma_+ + \lambda_-\gamma_-\,, \quad \alpha_3 = \lambda_+\gamma_+^2 + \lambda_-\gamma_-^2\,. \tag{3.10}$$

For $\alpha_2 \neq 0$, combining (3.8) with (3.9) we find

$$\begin{aligned}
d_e^2(\alpha_2^2 - \alpha_1\alpha_3)\nabla\times\nabla\times\nabla\times\mathbf{B} + (\alpha_3 + \alpha_1 d_e^2)\nabla\times\nabla\times\mathbf{B} \\
+(\alpha_2^2 - \alpha_1\alpha_3 - 1)\nabla\times\mathbf{B} + \alpha_1\mathbf{B} = 0\,,
\end{aligned} \tag{3.11}$$

which can be solved by a linear combination of Beltrami fields such as

$$\mathbf{B} = \sum_{j=1}^{3} c_j\boldsymbol{w}_j\,, \quad \nabla\times\boldsymbol{w}_j = \kappa_j\boldsymbol{w}_j\,, \quad j = 1,2,3\,. \tag{3.12}$$

Here, c_j are constants and the Beltrami parameters κ_j are solutions to the cubic equation

$$d_e^2(\alpha_2^2 - \alpha_1\alpha_3)\kappa^3 + (\alpha_3 + \alpha_1 d_e^2)\kappa^2 + (\alpha_2^2 - \alpha_1\alpha_3 - 1)\kappa + \alpha_1 = 0\,. \tag{3.13}$$

Similarly one can find a corresponding triple-Beltrami equation for the velocity field, or can express the velocity in terms of the magnetic field solution as follows

$$\mathbf{v} = \frac{\alpha_2^2 - \alpha_1\alpha_3}{\alpha_2}\mathbf{B}^* + \frac{\alpha_3}{\alpha_2}\nabla\times\mathbf{B}\,, \tag{3.14}$$

for $\alpha_2 \neq 0$. If the Lagrangian multipliers are chosen so as $\alpha_2 = 0$ then the system (3.8)–(3.9) decouples and the magnetic field equation can be solved by a double-Beltrami field while the velocity satisfies a single Beltrami condition. This singular behavior indicates that electron inertia can be the reason for the emergence of significantly different steady-state structures upon changing the generalized helicity content. Also, note that if $d_e = 0$ both $\mathbf{B}$ and $\mathbf{v}$ admit double-Beltrami solutions as it is well known within the HMHD context, e.g. see [78].

3.1.2 Equilibrium variational principle with helical symmetry

As mentioned in the previous chapter, the helically symmetric formulation includes both the translationally symmetric and axisymmetric cases being a more generic case for which a poloidal representation of the magnetic field is possible. By poloidal representation we mean a global description in terms of a component parallel to a symmetry direction and a flux function describing the field that lies on the plane perpendicular to this direction (poloidal plane), leading to configurations with well defined magnetic surfaces. In a series of papers this symmetry was employed for deriving equilibrium equations of the Grad-Shafranov type, i.e. PDEs with poloidal magnetic flux functions as dependent variables, [70, 79, 80, 81, 82, 83] in the context of standard MHD theory. Particularly in [70], the helically symmetric Grad-Shafranov or Johnson-Frieman-Kulsrud-Oberman (JFKO) (see [79, 81]) equation was derived using a Hamiltonian variational principle. The same approach is adopted also for our derivation, however, for the more complicated XMHD theory.

With the helically symmetric Casimirs at hand, we can build the EC variational principle to obtain equilibrium conditions. For analogous utilizations of this methodology for symmetric or 2D plasmas the reader is referred to [10, 17, 71, 84, 69, 70, 85, 86, 87]. As explained in Chapter 1, the EC principle states that phase space points that nullify the first variation EC functional $\mathcal{H}_C$ correspond to equilibria. In our case, requiring the vanishing of $\delta\mathcal{H}_C$ amounts to

$$\begin{aligned}\delta\int_D d^3x\ \ \Big\{&\rho\left(\frac{v_h^2}{2}+\frac{k^2}{2}|\boldsymbol{\nabla}\chi|^2+[\Upsilon,\chi]+\frac{|\nabla\Upsilon|^2}{2}+U(\rho)\right)\\ &+\frac{B_h^*B_h}{2}+\frac{k^2}{2}\nabla\psi^*\cdot\nabla\psi-(kB_h^*+\gamma\Omega-2n\ell\gamma k^3v_h)\mathcal{F}(\varphi)\\ &-2n\ell k^4\tilde{\mathcal{F}}(\varphi)-(kB_h^*+\mu\Omega-2n\ell\mu k^3v_h)\mathcal{G}(\xi)\\ &-2n\ell k^4\tilde{\mathcal{G}}(\xi)-\rho\mathcal{M}(\varphi)-\rho\mathcal{N}(\xi)\Big\}=0\,,\end{aligned} \tag{3.15}$$

where $\varphi := \psi^* + \gamma k^{-1} v_h$, $\xi := \psi^* + \mu k^{-1} v_h$ and recall that

$$\tilde{\mathcal{F}} := \int^{\varphi} \mathcal{F}(g)dg \quad \text{and} \quad \tilde{\mathcal{G}} := \int^{\xi} \mathcal{G}(g)dg\,. \tag{3.16}$$

Since the variations of the field variables are independent, (3.15) is satisfied if the coefficients of the field variable variations vanish. This requirement, upon exploiting the relations (2.41)–(2.46), leads to the following equilibrium conditions:

$$\begin{aligned}
\delta\rho \quad &: \quad [\rho U(\rho)]_\rho + \frac{|\mathbf{v}|^2}{2} - \mathcal{M}(\varphi) - \mathcal{N}(\xi) \\
&\quad + \frac{d_e^2}{2\rho^2}\left(J_h^2 + k^2|\nabla(k^{-1}B_h)|^2\right) = 0\,, && (3.17)\\
\delta\Upsilon \quad &: \quad -\nabla\cdot(\rho\nabla\Upsilon) + [\chi,\rho] = 0\,, && (3.18)\\
\delta\chi \quad &: \quad -\nabla\cdot(\rho k^2\nabla\chi) + [\rho,\Upsilon] - \gamma\mathcal{L}\mathcal{F}(\varphi) - \mu\mathcal{L}\mathcal{G}(\xi) = 0\,, && (3.19)\\
\delta v_h \quad &: \quad \rho v_h - \rho k^{-1}\left[\gamma\mathcal{M}'(\varphi) + \mu\mathcal{N}'(\xi)\right] - B_h^*\left[\gamma\mathcal{F}'(\varphi) + \mu\mathcal{G}'(\xi)\right] \\
&\quad - k^{-1}(\Omega - 2n\ell k^3 v_h)\left[\gamma^2\mathcal{F}'(\varphi) + \mu^2\mathcal{G}'(\xi)\right] = 0\,, && (3.20)\\
\delta B_h^* \quad &: \quad B_h - k\left[\mathcal{F}(\varphi) + \mathcal{G}(\xi)\right] = 0\,, && (3.21)\\
\delta\psi^* \quad &: \quad \mathcal{L}\psi - kB_h^*\left[\mathcal{F}'(\varphi) + \mathcal{G}'(\xi)\right] - 2n\ell k^4\left[\mathcal{F}(\varphi) + \mathcal{G}(\xi)\right] \\
&\quad - (\Omega - 2n\ell k^3 v_h)\left[\gamma\mathcal{F}'(\varphi) + \mu\mathcal{G}'(\xi)\right] - \rho\left[\mathcal{M}'(\varphi) + \mathcal{N}'(\xi)\right] = 0\,. && (3.22)
\end{aligned}$$

Note that the lhs of (3.17)–(3.22) are the coefficients of the variations $(\delta\rho, \delta\Upsilon, \delta\chi, \delta v_h, \delta B_h^*, \delta\psi^*)$ in $\delta\mathcal{H}_C$. In addition to these terms, some surface boundary terms emerged in $\delta\mathcal{H}_C$ due to integration by parts. We assumed that those terms vanish, something which is true if $\delta\Upsilon, \delta\chi, \delta\psi^*$ vanish on the boundary ∂D. Equation (3.17) represents a Bernoulli law

$$\tilde{p}(\rho) \;=\; \rho\left[\mathcal{M}(\varphi) + \mathcal{N}(\xi)\right] - \rho\frac{|\mathbf{v}|^2}{2} - \frac{d_e^2}{2\rho}\left[J_h^2 + k^2|\nabla(k^{-1}B_h)|^2\right]\,, \tag{3.23}$$

where $\tilde{p} := \rho[\rho U(\rho)]_\rho = \rho h(\rho)$ where $h(\rho)$ is the total specific enthalpy ($\tilde{p} = \Gamma p/(\Gamma - 1)$ if we adopt the equation of state $p \propto \rho^\Gamma$ with Γ being the adiabatic constant). Equation (3.23) describes the effect of macroscopic equilibrium flow including the electron inertial effects (expressed via the magnetic terms), in the total pressure.

3.2 The JFKO-Bernoulli system

System (3.17)–(3.22) can be cast into a JFKO-Bernoulli PDE form that describes completely helically symmetric XMHD equilibria. This can be done by exploiting (3.18), (3.19), (3.21) and (2.35) in order to turn (3.20) and (3.22) into a coupled system for the flux functions φ and ξ. These equations, except of their coupling to Bernoulli equation, are additionally coupled to the definition (2.36) given in terms of

φ and ξ, which essentially is the helical component of Ampere's law. The derivation of this system requires some tedious manipulations, whose main steps are presented here. A good starting point is to observe that (3.18) and (3.19) can be written as

$$\mathbf{h}\cdot\nabla\times\boldsymbol{Q}=0 \quad \text{and} \quad \nabla\cdot(k^2\boldsymbol{Q})=0\,, \tag{3.24}$$

respectively, with

$$\boldsymbol{Q}:=\rho\nabla\chi-\rho k^{-2}\nabla\Upsilon\times\mathbf{h}-\gamma\nabla\mathcal{F}-\mu\nabla\mathcal{G}\,. \tag{3.25}$$

Therefore, the mutual solution of (3.18) and (3.19), should satisfy

$$\rho\nabla\chi-\rho k^{-2}\nabla\Upsilon\times\mathbf{h}=\gamma\nabla\mathcal{F}+\mu\nabla\mathcal{G}\,, \tag{3.26}$$

by which we can easily deduce

$$\Omega=-\gamma\nabla\cdot\left(\rho^{-1}k^2\mathcal{F}'\nabla\varphi\right)-\mu\nabla\cdot\left(\rho^{-1}k^2\mathcal{G}'\nabla\xi\right)\,. \tag{3.27}$$

Now, Eq. (3.27) can be inserted into (3.20) resulting in

$$\begin{aligned}\rho v_h+2n\ell k^2 v_h\left(\gamma^2\mathcal{F}'+\mu^2\mathcal{G}'\right)&=B_h^*\left(\gamma\mathcal{F}'+\mu\mathcal{G}'\right)+\rho k^{-1}\left(\gamma\mathcal{M}'+\mu\mathcal{N}'\right)\\ &-k^{-1}\left(\gamma^2\mathcal{F}'+\mu^2\mathcal{G}'\right)\left[\gamma\nabla\cdot\left(\frac{k^2\mathcal{F}'}{\rho}\nabla\varphi\right)+\mu\nabla\cdot\left(\frac{k^2\mathcal{G}'}{\rho}\nabla\xi\right)\right]\,.\end{aligned} \tag{3.28}$$

Note that the helical component of the flow can be easily expressed in terms of φ and ξ as

$$v_h=k\frac{\varphi-\xi}{\gamma-\mu}\,, \tag{3.29}$$

while for the helical component of the generalized magnetic field we have to invoke Eqs. (2.35) and (3.21) to write

$$B_h^*=(1+\varsigma)k(\mathcal{F}+\mathcal{G})-d_e^2k^{-1}\nabla\left[\rho^{-1}k^2\nabla(\mathcal{F}+\mathcal{G})\right]-2n\ell d_e^2\rho^{-1}k\mathcal{L}\psi\,, \tag{3.30}$$

where $\varsigma:=4n^2\ell^2d_e^2\rho^{-1}k^4$. Substituting Eqs. (3.29), (3.30) into Eq. (3.28) and after some manipulations, we end up with

$$\begin{aligned}&k^2\frac{\varphi-\xi}{\gamma-\mu}\left[\rho+2n\ell k^2(\gamma^2\mathcal{F}'+\mu^2\mathcal{G}')\right]=\rho(\gamma\mathcal{M}'+\mu\mathcal{N}')\\ &+(1+\varsigma)k^2(\mathcal{F}+\mathcal{G})(\gamma\mathcal{F}'+\mu\mathcal{G}')-2n\ell d_e^2\rho^{-1}k^2(\gamma\mathcal{F}'+\mu\mathcal{G}')\mathcal{L}\psi\\ &-\gamma(\gamma^2+d_e^2)\mathcal{F}'\nabla\cdot\left(\frac{k^2}{\rho}\nabla\mathcal{F}\right)-\mu(\mu^2+d_e^2)\mathcal{G}'\nabla\cdot\left(\frac{k^2}{\rho}\nabla\mathcal{G}\right)\,,\end{aligned} \tag{3.31}$$

where $\gamma\mu = -d_e^2$ has been used. Now, upon inserting (3.27), (3.30) and (3.29) into (3.22) we can find

$$\begin{aligned}
&\left[1 + 2n\ell d_e^2\rho^{-1}k^2(\mathcal{F}' + \mathcal{G}')\right]\mathcal{L}\psi + 2n\ell k^4\frac{\varphi - \xi}{\gamma - \mu}(\gamma\mathcal{F}' + \mu\mathcal{G}') \\
&= (1 + \varsigma)k^2(\mathcal{F} + \mathcal{G})(\mathcal{F}' + \mathcal{G}') + 2n\ell k^4(\mathcal{F} + \mathcal{G}) + \rho(\mathcal{M}' + \mathcal{N}') \\
&\quad -(\gamma^2 + d_e^2)\mathcal{F}'\nabla\cdot\left(\frac{k^2}{\rho}\nabla\mathcal{F}\right) - (\mu^2 + d_e^2)\mathcal{G}'\nabla\cdot\left(\frac{k^2}{\rho}\nabla\mathcal{G}\right). \qquad (3.32)
\end{aligned}$$

Equations (3.31) and (3.32) can be combined to a system for which within each equation appears the differential of φ or ξ only; by doing so, after some algebra we find

$$\begin{aligned}
&(\gamma^2 + d_e^2)\mathcal{F}'\nabla\cdot\left(\frac{k^2}{\rho}\nabla\mathcal{F}\right) = (1 + \varsigma)k^2(\mathcal{F} + \mathcal{G})\mathcal{F}' + \left(\frac{\mu}{\gamma - \mu} - 2n\ell\frac{d_e^2}{\rho}k^2\mathcal{F}'\right)\mathcal{L}\psi \\
&+\rho\mathcal{M}' - 2n\ell\frac{\mu}{\gamma - \mu}k^4(\mathcal{F} + \mathcal{G}) - k^2\left[\frac{\rho}{(\gamma - \mu)^2} + 2n\ell\frac{\gamma}{\gamma - \mu}k^2\mathcal{F}'\right](\varphi - \xi), \quad (3.33) \\
&(\mu^2 + d_e^2)\mathcal{G}'\nabla\cdot\left(\frac{k^2}{\rho}\nabla\mathcal{G}\right) = (1 + \varsigma)k^2(\mathcal{F} + \mathcal{G})\mathcal{G}' - \left(\frac{\gamma}{\gamma - \mu} + 2n\ell\frac{d_e^2}{\rho}k^2\mathcal{G}'\right)\mathcal{L}\psi \\
&+\rho\mathcal{N}' + 2n\ell\frac{\gamma}{\gamma - \mu}k^4(\mathcal{F} + \mathcal{G}) + k^2\left[\frac{\rho}{(\gamma - \mu)^2} - 2n\ell\frac{\mu}{\gamma - \mu}k^2\mathcal{G}'\right](\varphi - \xi). \quad (3.34)
\end{aligned}$$

To close the system, let us consider the definitions of φ and ξ and also Eqs. (2.36) and (3.21). All these can be combined to give

$$\mathcal{L}\psi = k^2\frac{\rho}{d_e^2}\left[\frac{\mu\varphi - \gamma\xi}{\mu - \gamma} - \psi + 2n\ell d_e^2\rho^{-1}k^2(\mathcal{F} + \mathcal{G})\right]. \qquad (3.35)$$

Equations (3.33)–(3.35) coupled to Bernoulli equation (3.23) describe completely the equilibria in terms of the flux functions ψ, φ, ξ and mass density ρ, for given free functions $\mathcal{F}(\varphi)$, $\mathcal{G}(\xi)$, $\mathcal{M}(\varphi)$, $\mathcal{N}(\xi)$ and a thermodynamic closure $p = p(\rho)$, since all physical quantities of interest can be expressed in terms of ψ, φ, ξ and ρ. Namely, the helical component of the magnetic field is given by (3.21), the poloidal field is simply $\nabla\psi \times \mathbf{h}$; the helical component of the velocity is given by (3.29) and for the corresponding poloidal component we take the cross product of (3.26) with $\mathbf{h}$, to obtain

$$\mathbf{v}_p = \rho^{-1}(\gamma\nabla\mathcal{F} + \mu\nabla\mathcal{G}) \times \mathbf{h}. \qquad (3.36)$$

Due to the three coupled PDEs, which have to be solved simultaneously, and the imposed helical symmetry that inserts some additional terms because of the nonorthogonality of the basis vectors, the solution of this system in conjunction with Bernoulli relation (3.23), consists a rather complicated problem. For this reason we present below special cases of equilibria including, axisymmetric XMHD and HMHD, incompressible XMHD and barotropic and incompressible Hall MHD equilibria with helical symmetry presenting the corresponding system of Grad-Shafranov or JFKO equations for each of the aforementioned cases.

3.2.1 Hall MHD equilibria

The Hall MHD limit is effected by setting $d_e = 0$ and thereby neglecting electron inertial effects. Thus, $\gamma = d_i$, $\mu = 0$, and the flux functions become $\varphi = \psi + d_i k^{-1} v_h$ and $\xi = \psi$. In this model, only ion drift effects are considered and the electron surfaces coincide with the magnetic ones. The JFKO system for computing the poloidal ion and magnetic fluxes is

$$d_i^2 \mathcal{F}' \nabla \cdot \left(\frac{k^2}{\rho} \nabla \mathcal{F} \right) = k^2 (\mathcal{F} + \mathcal{G}) \mathcal{F}' + \rho \mathcal{M}' - k^2 \left[\frac{\rho}{d_i^2} + 2n\ell k^2 \mathcal{F}' \right] (\varphi - \psi) \, , \tag{3.37}$$

$$\mathcal{L}\psi = k^2 (\mathcal{F} + \mathcal{G}) \mathcal{G}' + \rho \mathcal{N}' + 2n\ell k^4 (\mathcal{F} + \mathcal{G}) + k^2 \rho \frac{(\varphi - \psi)}{d_i^2} \, . \tag{3.38}$$

Helically symmetric Hall MHD equilibria are completely determined by the above equations coupled to a Bernoulli law, which can be deduced from (3.23) for $d_e = 0$, allowing for the self-consistent computation of the mass density ρ for a given equation of state $p = p(\rho) \propto \rho^\Gamma$. The HMHD Bernoulli equation is

$$\frac{\Gamma p}{\Gamma - 1} = \rho \left[\mathcal{M} + \mathcal{N} - k^2 \frac{(\varphi - \psi)^2}{2d_i^2} \right] - d_i^2 k^2 \frac{(\mathcal{F}')^2}{2\rho} |\nabla \varphi|^2 \, . \tag{3.39}$$

Also from (3.29) and (3.36) we have

$$v_h = k \frac{\varphi - \psi}{d_i} \quad \text{and} \quad \mathbf{v}_p = d_i \frac{\mathcal{F}'}{\rho} \nabla \varphi \times \mathbf{h} \, . \tag{3.40}$$

For $\ell = 0$, (3.37), (3.38) and (3.39) reduce to the axisymmetric Grad-Shafranov-Bernoulli system of [88] that is presented below. For the baroclinic version of the axisymmetric HMHD equilibrium equations the reader is referred to [89, 90].

3.2.2 Axisymmetric barotropic XMHD and HMHD

The axisymmetric equilibrium equations are obtained by setting the helical angle a to zero, i.e., $\ell = 0$ and $n = -1$, so the parameter ς is zero and the scale factor $k = 1/r$ and $\mathbf{h} = r^{-1} \hat{e}_\phi$. With these parameters, (3.33)–(3.35) reduce to the following Grad-Shafranov system:

$$(\gamma^2 + d_e^2) \mathcal{F}' r^2 \nabla \cdot \left(\frac{\mathcal{F}'}{\rho} \frac{\nabla \varphi}{r^2} \right) = \mathcal{F}' (\mathcal{F} + \mathcal{G}) + r^2 \rho \mathcal{M}' - \frac{\mu}{\gamma - \mu} \Delta^* \psi - \rho \frac{\varphi - \xi}{(\gamma - \mu)^2} \, , \tag{3.41}$$

$$(\mu^2 + d_e^2) \mathcal{G}' r^2 \nabla \cdot \left(\frac{\mathcal{G}'}{\rho} \frac{\nabla \xi}{r^2} \right) = \mathcal{G}' (\mathcal{F} + \mathcal{G}) + r^2 \rho \mathcal{N}' + \frac{\gamma}{\gamma - \mu} \Delta^* \psi + \rho \frac{\varphi - \xi}{(\gamma - \mu)^2} \, , \tag{3.42}$$

$$\Delta^* \psi = \frac{\rho}{d_e^2} \left(\psi - \frac{\mu \varphi - \gamma \xi}{\mu - \gamma} \right) \, , \tag{3.43}$$

where $\Delta^* := r^2\nabla\cdot\left(\nabla/r^2\right)$ is the so-called Shafranov operator. Equation (3.23) assumes the form:

$$\frac{\Gamma p}{\Gamma-1} = \rho\left[\mathcal{M}(\varphi)+\mathcal{N}(\xi)\right]-\rho\frac{|\mathbf{v}|^2}{2}-\frac{d_e^2}{2\rho}\left[J_\phi^2+r^{-2}|\nabla(rB_\phi)|^2\right], \tag{3.44}$$

where $J_\phi = -r^{-1}\Delta^*\psi$ is the toroidal current density. For $d_e = 0$ we obtain the axisymmetric Hall MHD Grad-Shafranov-Bernoulli system [88], which reads as follows

$$d_i^2\mathcal{F}'r^2\nabla\cdot\left(\frac{\mathcal{F}'}{\rho}\frac{\nabla\varphi}{r^2}\right)=\mathcal{F}'(\mathcal{F}+\mathcal{G})+r^2\rho\mathcal{M}'-\rho\frac{\varphi-\psi}{d_i^2}, \tag{3.45}$$

$$\Delta^*\psi+\mathcal{G}'(\mathcal{F}+\mathcal{G})+r^2\rho\mathcal{N}'+\rho\frac{\varphi-\psi}{d_i^2}=0, \tag{3.46}$$

$$\rho h(\rho)=\frac{\Gamma p}{\Gamma-1}=\rho\left[\mathcal{M}(\varphi)+\mathcal{N}(\psi)\right]-\rho\frac{(\varphi-\psi)^2}{2d_i^2r^2}-d_i^2\frac{(\mathcal{F}')^2}{2\rho r^2}|\nabla\varphi|^2, \tag{3.47}$$

3.2.3 Numerical Hall MHD equilibria

In this subsection the axisymmetric barotropic Hall MHD equilibrium system is solved numerically. It should be stressed here that in order to perform this computation, consulting previous studies concerned with the numerical integration of two-fluid equilibrium systems such as [90, 91, 92] was particularly instructive. Our computation is realized using a Successive Over-Relaxation (SOR) iterative solver with Red-Black ordering and Chebyshev acceleration for faster convergence. The grid $N_r \times N_z$ is uniform with different spacing in r and z directions. Brent's method is utilized to find the roots of the Bernoulli equation. Brent's algorithm is a hybrid, failsafe method that combines inverse quadratic interpolation with the Secant and bisection methods to determine bracketed roots. For further information on the aforementioned methods one can consult [93] and also Appendix A. To apply Brent's algorithm we need somehow to bracket the root within an interval. To do so we first evaluate two characteristic density values, one corresponding to the maximum possible root and the other to the local minimum of the Bernoulli function

$$b(\rho)=\frac{p_1\Gamma}{\Gamma-1}\rho^{\Gamma-1}-b_1+\frac{b_2}{\rho^2}, \tag{3.48}$$

where $b_1=\mathcal{M}(\varphi)+\mathcal{N}(\psi)-v_\phi^2/2$ and $b_2=d_i^2(\mathcal{F}'(\varphi))^2|\nabla\varphi|^2/2$, the specific enthalpy term results from $p(\rho)=p_1\rho^\Gamma$ and we choose $\Gamma=5/3$. Bernoulli equation is given by $b(\rho)=0$. Necessarily $b_1>0$ because otherwise the pressure will be negative due to the positiveness of the last term in (3.48). Within the domain $(0,\infty)$, $b(\rho)$ has a global minimum if $b_2\neq 0$, which corresponds to a density ρ_e evaluated upon solving the following equation

$$db(\rho)/d\rho=p_1\Gamma\rho^{\Gamma-2}-\frac{2b_2}{\rho^3}=0. \tag{3.49}$$

Therefore, $b(\rho)$ possess from zero up to two real roots. To assess that real roots exist we evaluate $b(\rho_e)$ which has to be zero or negative. In the former case ρ_e is the single root whereas in the latter there exist two roots and then one limit of the bracketing interval will be ρ_e. If ρ_e is the upper limit then Brent's method will evaluate the "lighter" (lower) root of $b(\rho)$. To find the "heavier" (larger) root we need a ρ_{max} which is the maximum possible root of the Bernoulli function. This can be found considering $\rho \gg 1$. In this limit the last term in (3.48) is negligible and therefore we can directly solve for ρ to find

$$\rho_{max} = \left(\frac{\Gamma - 1}{p_1 \Gamma} b_1\right)^{1/(\Gamma-1)} . \tag{3.50}$$

Therefore, the heavier root is bracketed within the search interval $[\rho_e, \rho_{max}]$. Both bracketing options can be considered, however, in the particular example presented below, the heavier root is used to represent the mass density at the interior points, while for the boundary points, $\rho = 0$ is imposed. Note that mathematically ρ is allowed to go to zero because b_2 is selected to vanish on the boundary.

For the solution of Eqs. (3.45), (3.46), following the references [90, 91, 92], we start solving Eq. (3.46) for ψ iteratively, i.e. the next approximation $\psi^{(n+1)}$ is obtained using the previous approximations $\psi^{(n)}$, $\varphi^{(n)}$, and $\rho^{(n)}$. The initial condition for ψ, i.e. $\psi^{(0)}$ is taken to be the corresponding static HMHD equilibrium, obtained upon solving Eq. (3.46) with $\varphi = \psi$, therefore $\varphi^{(0)} = \psi^{(0)}$. Knowing $\psi^{(n+1)}$ and $\varphi^{(n)}$ we solve Eq. (3.47) for $\rho^{(n+1)}$ and then Eq. (3.45) is considered as an algebraic equation for $(\varphi - \psi)^{(n+1)}$ which allows the evaluation of the next approximation for φ. The procedure is repeated until the residual error of (3.46) and also the maximum difference from the previous computed values, are smaller than predefined tolerances. For more details the reader is referred to Appendix A.

Another feature incorporated to this solver is that it allows the computation on up-down poloidally asymmetric domains with diverted boundaries having a lower x-point, which are prescribed analytically. Prescription of the boundary is made upon using the analytic formulas of [94]. More specifically, the boundary is described by

$$r = 1 + \epsilon_0 \cos(\tau + \sin^{-1}(\delta_u)\sin(\tau)) , \tag{3.51}$$

$$z = k_u \epsilon_0 \sin(\tau) , \tag{3.52}$$

for its upper part ($z > 0$) where $\epsilon_0 = a_0/R_0$ is the inverse aspect ratio of the torus with a_0 and R_0 being the minor and major radii, respectively. Also, δ_u and k_u are the upper triangularity and elongation of the poloidal cross section, respectively. Parameter τ is expressed in terms of the poloidal angle θ as

$$\tau(\theta) = \tau_0 \theta^2 + \tau_1 \theta^n ,$$

$$\begin{aligned}
\tau_0 &= \frac{\theta_{\delta_u}^n - 0.5\pi^n}{\pi\theta_{\delta_u}^n - \theta_{\delta_u}^2\pi^{n-1}}\,,\\
\tau_1 &= \frac{-\theta_{\delta}^2 + 0.5\pi^2}{\pi\theta_{\delta_u}^n - \theta_{\delta_u}^2\pi^{n-1}}\,,\\
\theta_{\delta_u} &= \pi - \tan^{-1}(k_u/\delta_u)\,. \qquad (3.53)
\end{aligned}$$

The lower part is given by the following formulas

$$\begin{aligned}
r &= 1 + \epsilon_0\cos(\theta)\,,\\
z &= -[2q_1\epsilon_0(1+\cos(\theta))]\,.\\
q_1 &= \frac{(k_d\epsilon_0)^2}{2\epsilon_0(1+\cos\theta_{\delta_d})}\,, \quad \pi \le \theta \le 2\pi - \theta_{\delta_d}\,, \qquad (3.54)
\end{aligned}$$

for the left part of the boundary and

$$\begin{aligned}
r &= 1 + \epsilon_0\cos(\theta)\,,\\
z &= -\sqrt{2q_2\epsilon_0(1-\cos(\theta))}\,.\\
q_2 &= \frac{(k_d\epsilon_0)^2}{2\epsilon_0(1-\cos\theta_{\delta_d})}\,, \quad 2\pi - \theta_{\delta_d} \le \theta \le 2\pi\,, \qquad (3.55)
\end{aligned}$$

for the right part of the boundary. The construction of the computational boundary is based on the identification of the grid points that are nearest neighbors to the analytic boundary curve. We take the boundary condition on these points to be $\psi = \varphi = 0$. The particular class of equilibria considered here corresponds to the following ansatz for the free functions $\mathcal{F}$, $\mathcal{G}$ $\mathcal{M}$:

$$\begin{aligned}
\mathcal{F} &= \quad f_0 + f_1\varphi + \frac{1}{2}f_2\varphi^2 + \frac{1}{3}f_3\varphi^3\,,\\
\mathcal{G} &= \quad g_0 + g_1\psi + \frac{1}{2}g_2\psi^2 + \frac{1}{3}g_3\psi^3\,,\\
\mathcal{M} &= \quad m_0 + m_1\varphi + \frac{1}{2}m_2\varphi^2 + \frac{1}{3}m_3\varphi^3\,, \qquad (3.56)
\end{aligned}$$

with $f_1 = 0$ because we want the poloidal velocity to vanish on the boundary. For the free function $\mathcal{N}$ we make a different choice which allows the computation of equilibria with mass density and pressure pedestal, namely we assume

$$\mathcal{N}(\psi) = (n_0 + n_1\psi^2)\left(1 - e^{-\psi^2/n_2}\right)\,. \qquad (3.57)$$

However, for the computation of the initial static equilibrium we used a polynomial ansatz for $\mathcal{N}$, which results in a typical pressure profile without pedestal. That is, our computation starts from an L-mode equilibrium without flow and iteratively relaxes to an H-mode-like equilibrium with mass density and pressure pedestal, steep gradients and localized sheared flows.

For the equilibrium presented in Fig. 3.1 the free parameters appearing in (3.56)–(3.57) are selected so as the values of the characteristic physical quantities to be consistent with the experimental results in big Tokamaks and also with what is expected for the International Thermonuclear Reactor (ITER). Also, the parameters appearing in (3.51)–(3.55) are selected in connection with the ITER design. Specific details on this particular equilibrium are presented in Figs. 3.1–3.7. The cylindrical coordinates r, z are normalized with respect to the major radius $R_0 = 6.2\ m$ i.e. $r = R/R_0$ and $z = Z/R_0$ where R, Z are the dimensional coordinates in connection with the poloidal plane. The dimensions in various dimensional quantities are restored upon multiplying with the corresponding Alfvén normalization constants. Also, in table **??** the main geometric parameters, characteristic figures of merit and values of various equilibrium quantities are summarized. In addition, two figures of merit for the numerical computation are presented in Fig. 3.8; these are the maximum local convergence rate and the maximum local residual error of ψ in each iteration. Both converge to sufficiently small values within the first 300 iterations and then they exhibit a slightly convergent-stagnation behavior for at least 200 iterations, which indicates that the algorithm reached to a "good" numerical solution. The computational domain was created on a 200×200 grid and the SOR parameter was relaxed by the Chebyshev procedure to the value 1.969.

The equilibrium results show that the Hall contribution has a small influence on the pressure, the mass and current density, and magnetic field profiles. However, as becomes evident from Figs. 3.3 this contribution strongly affects the flow profiles. Also, it is responsible for the separation of the magnetic and ion surfaces as shown in Fig. 3.9. These are the level sets of ψ and φ, respectively. Note, that this separation does not imply violation of quasineutrality, it merely means that two nearby fluid elements, an ionic and an electronic one, follow different paths lying on different surfaces as they travel within the plasma. We understand that this has consequences when it comes to the study of transport phenomena and therefore Hall MHD equilibria or even better XMHD equilibria should be preferred over MHD equilibria when further investigations involving such micro-motions are going to be performed. One of the aims of this thesis is to delineate the employment of a concise and powerful methodology, i.e. the EC principle, in order to obtain useful equilibrium equations for models that contain such physics, no matter how complicated the equations of motion appear to be. Apart from the results presented here, with specific values of d_i i.e. $d_i = 0.01$ and $d_i = 0.05$, we computed also equilibria with values up to $d_i = 0.20$ and corroborated that the various profiles change accordingly, in a way consistent with the results shown in figures 3.2–3.7.

The flow and current density profiles are consistent with the high mode phenomenology, where highly sheared flows and high bootstrap current, caused due to strong pressure gradients, are observed in the edge transport barrier region. In the

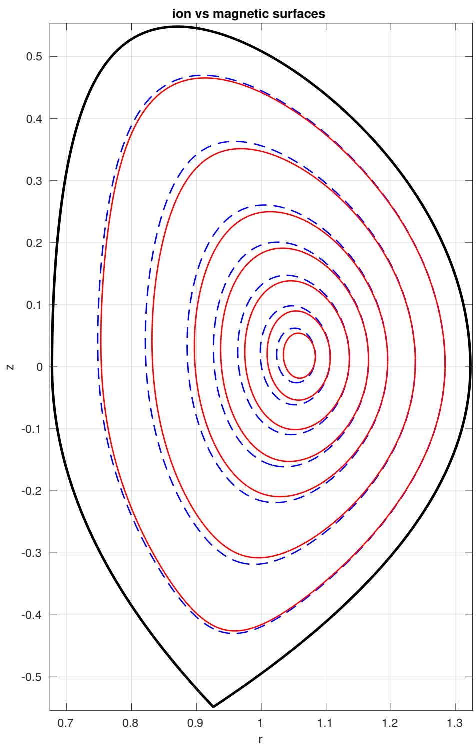


FIGURE 3.1: The poloidal cross section of ion (dashed blue) and magnetic surfaces (solid red) for a numerical, ITER-like equilibrium, in connection with ansatz (3.56), (3.57) and Hall parameter $d_i = 0.05$.

table below some characteristic geometric parameters and values of physical quantities of interest are summarized. Note that the speed of sound is one order of magnitude larger than the flow velocity all over the plasma volume, hence the equilibrium problem is elliptic (see Section 3.4).

minor radius a (m)	2.0
inverse aspect ratio ϵ_0	0.32
elongation $k_u = k_d$	1.70
triangularity $\delta_u = \delta_d$	0.40
β_{max}	0.031
Norm. Shafranov shift Δ_s	0.062
$J_{t_{max}}$ (A/m^2)	1.519×10^6
J_{t_a} (A/m^2)	2.720×10^5
p_{max} (Pa)	1.216×10^5
$\lvert v_t \rvert_{max}$ (m/s)	1.596×10^5
$v_{p_{max}}$ (m/s)	7.026×10^3
$c_{s_{max}}$ (m/s)	2.489×10^6

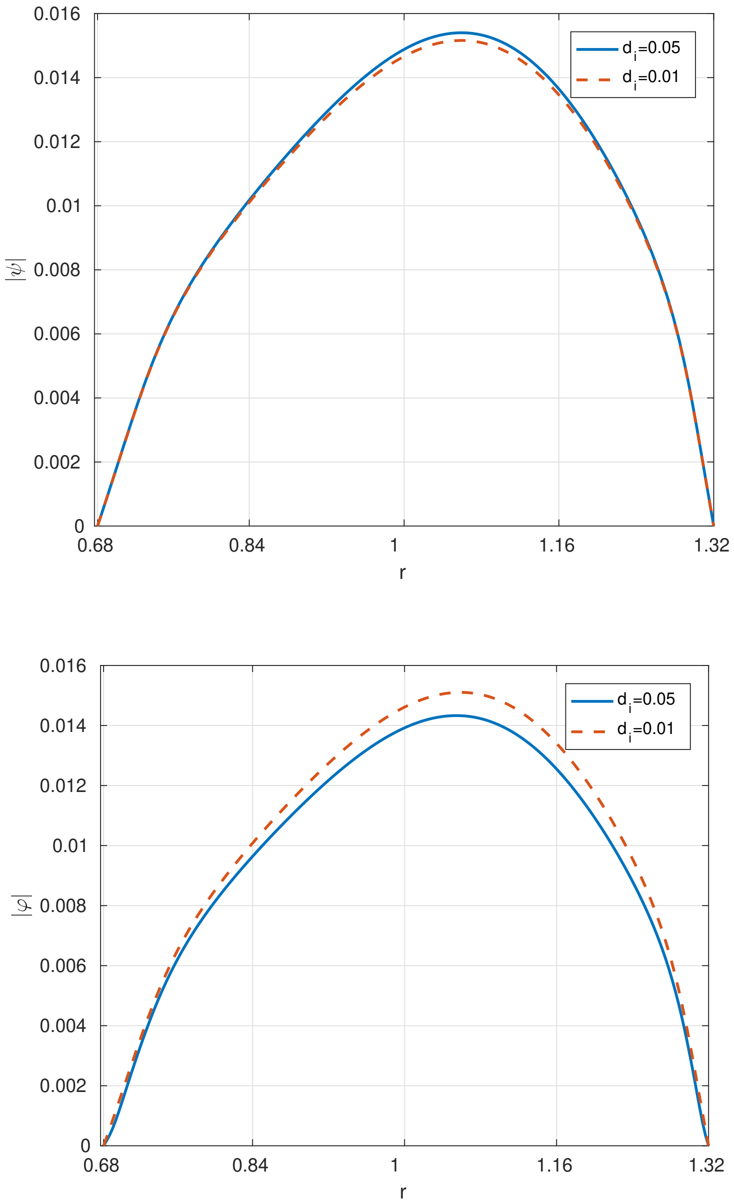


FIGURE 3.2: Up: the magnetic flux function ψ on the plane $z = z_a$, where z_a corresponds to magnetic axis, for two different values of the Hall parameter. Down: the corresponding diagrams for the ion stream function φ.

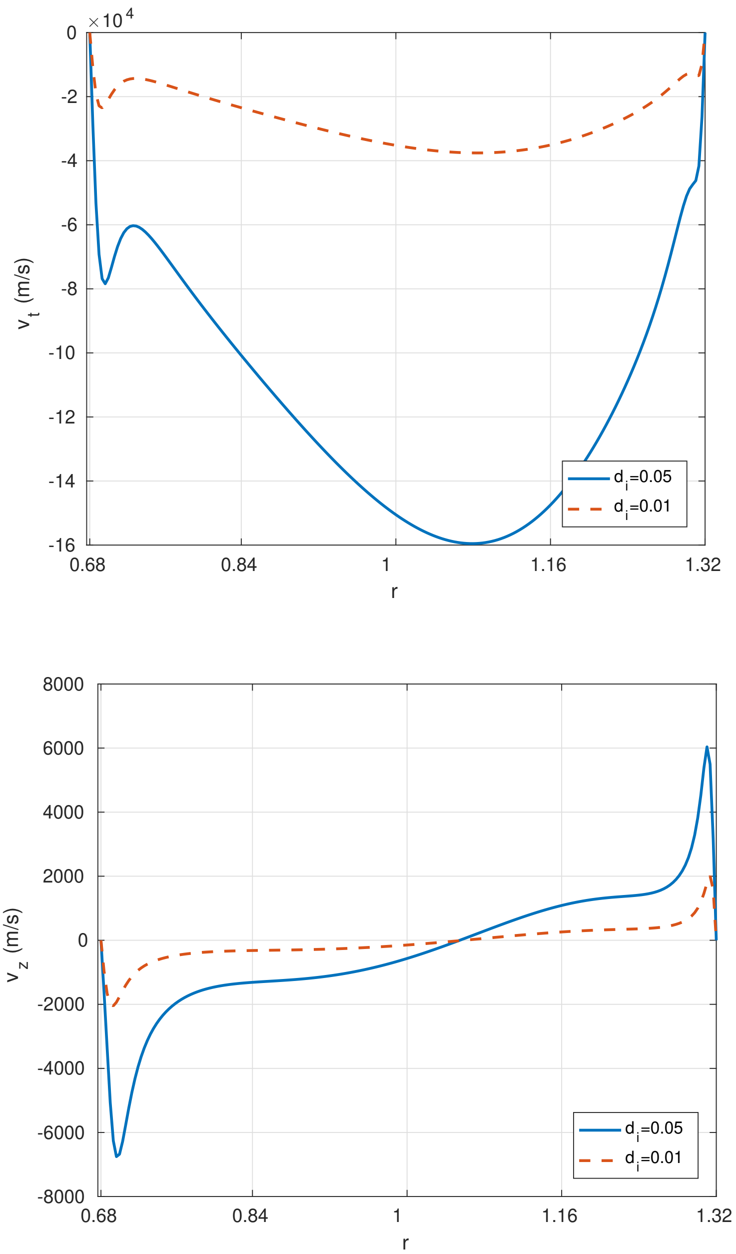


FIGURE 3.3: Up: the toroidal velocity field on the plane $z = z_a$ for $d_i = 0.01$ and $d_i = 0.05$. Down: the corresponding diagrams for the z component of the velocity field. We observe that strong flow shear accumulated towards the boundary, in the edge transport barrier region.

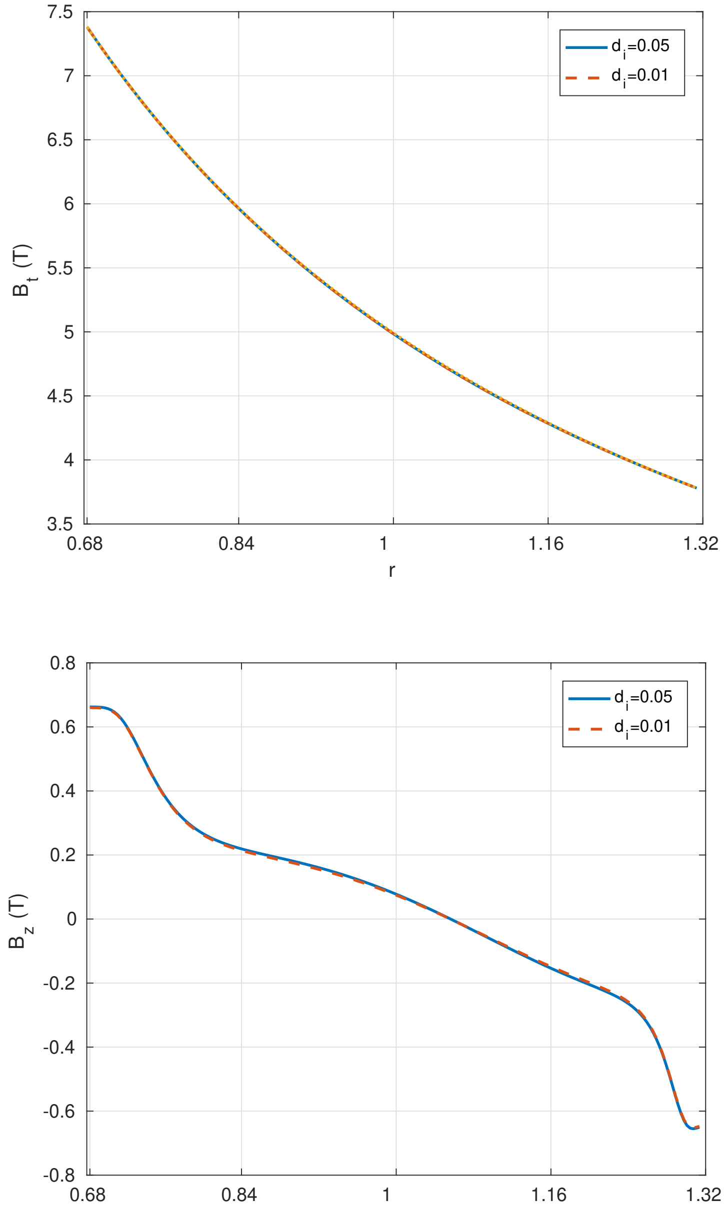
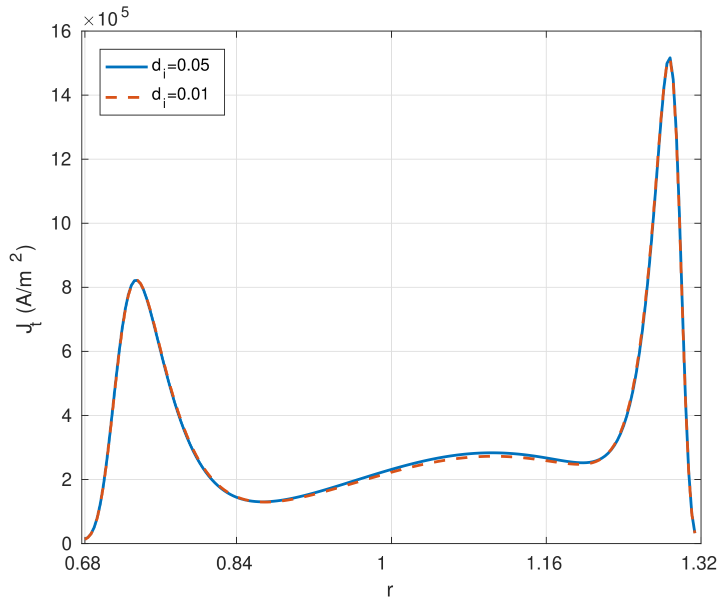


FIGURE 3.4: Up: the toroidal magnetic field is slightly lower than the vacuum field in the plasma core, revealing a diamagnetic behavior. Down: the z component of the magnetic field on $z = z_a$.



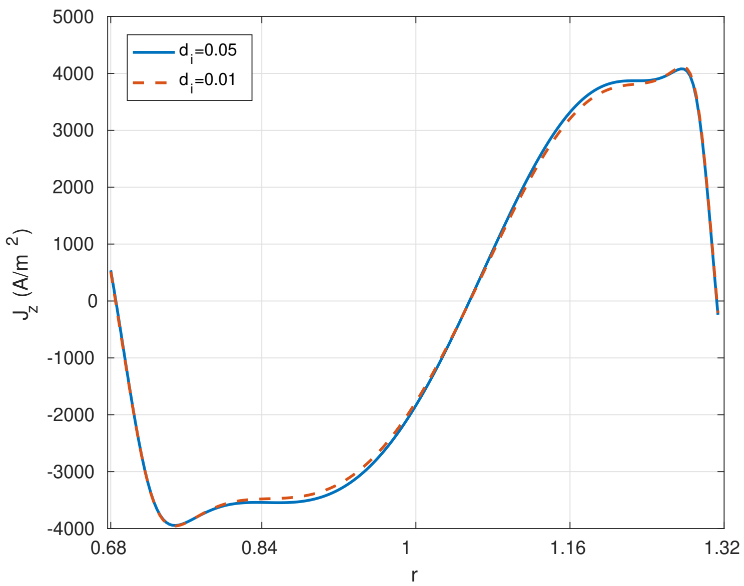
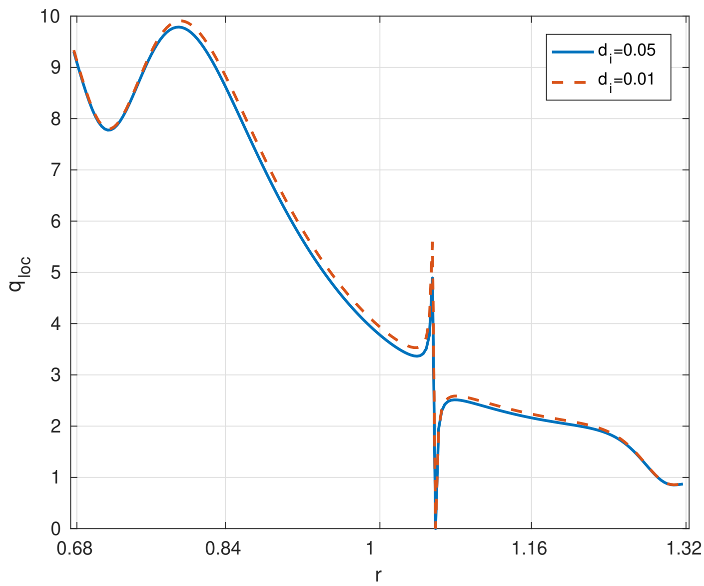


FIGURE 3.5: Up: the toroidal current density profile showing that strong currents are accumulated in the transport barrier region. This behavior is typical for plasmas with edge bootstrap current. Down: the z component of the current density on the plane $z = z_a$. The influence of d_i is very small.



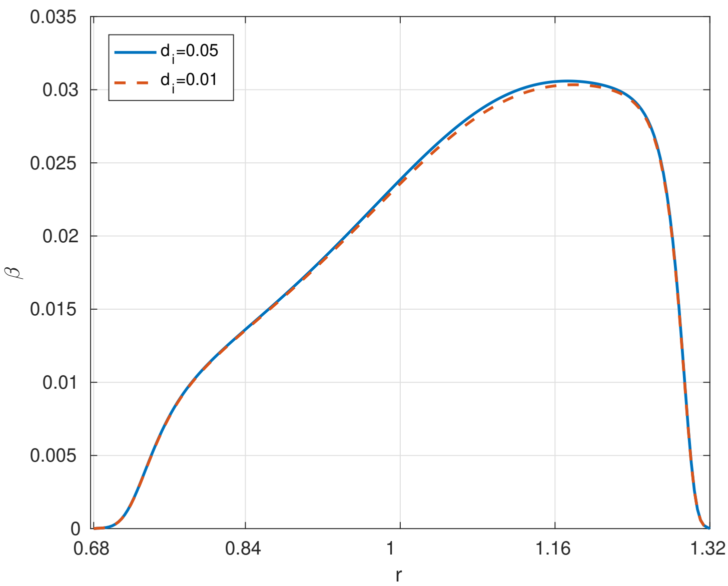


FIGURE 3.6: Up: the profile of the local safety factor $q_{loc} = r_s B_t/(rB_p)$ (where r_s is the distance from the magnetic axis). Down: the total β parameter defined as $\beta := 2\mu_0 P/B^2$.

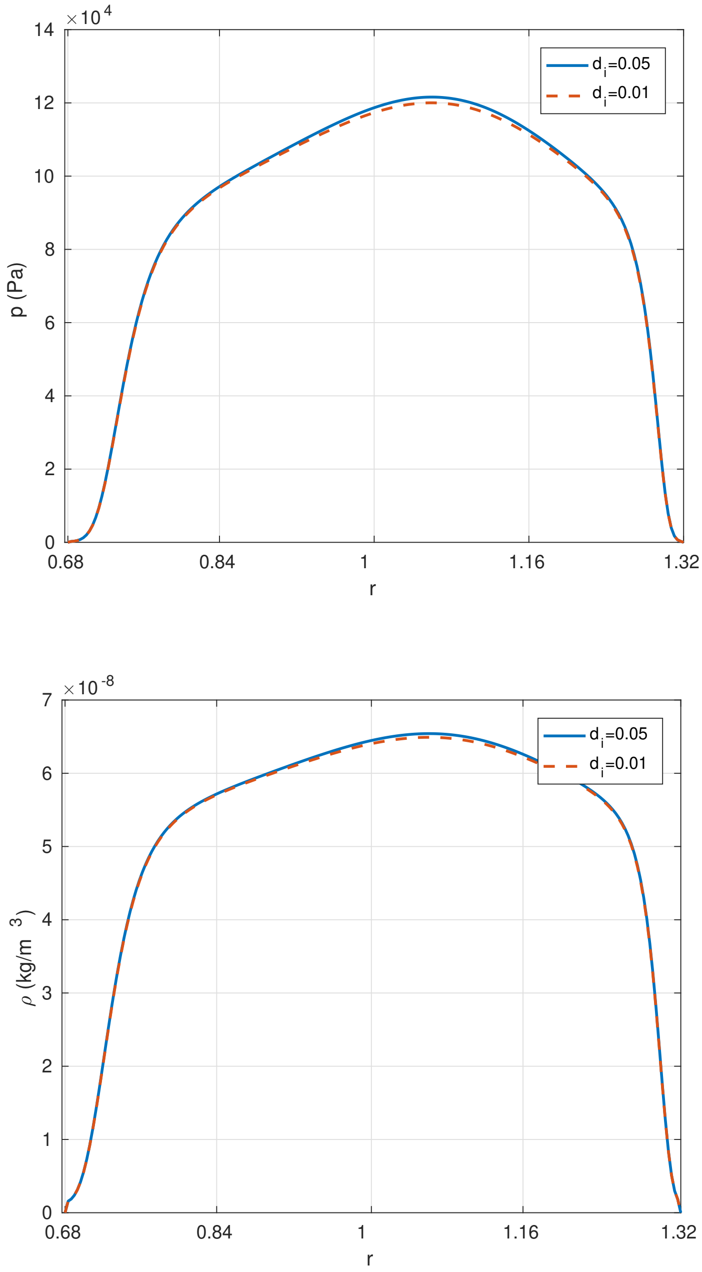


FIGURE 3.7: Up: the pressure profile on $z = z_a$ exhibits the formation of pedestal and steep gradients associated with H-mode operation. Down: the mass density profile, having analogous behavior.

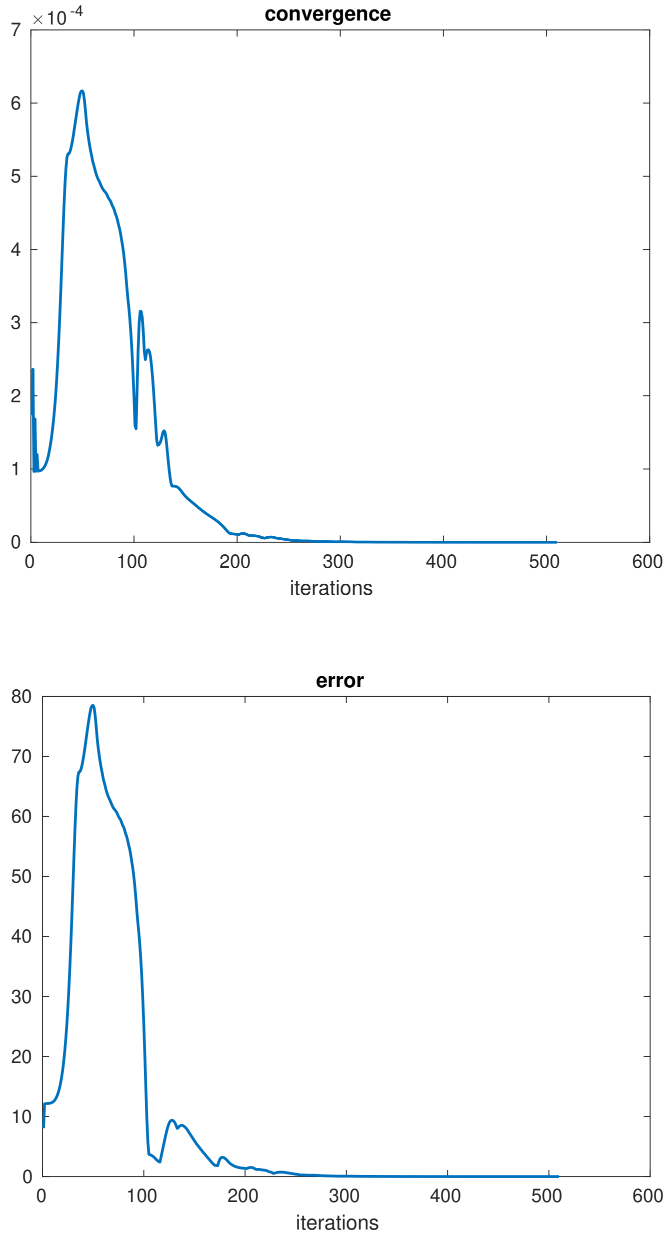


FIGURE 3.8: Up: The convergence diagram showing the behavior of $max(max(\psi^{(n+1)} - \psi^{(n)}), max(\varphi^{(n+1)} - \varphi^{(n)}))$. Down: the maximum residual error for ψ. The optimal relaxation parameter for a 200×200 grid was found to be $\omega = 1.969$.

3.3 Incompressible equilibria

To obtain the equilibrium system for incompressible plasmas with uniform mass density, we set $\rho = 1$. Note that incompressibility may refer also to the kind of the flows, that is, flows with divergence-free velocity fields that renders the mass density a Lagrangian invariant, which means that ρ is advected by the flow. Here, we address the simpler case where the mass density is constant. One should be careful when adopting this assumption because it has to be imposed a priori, i.e., before varying the EC functional. This is because, if we use the barotropic version of the EC functional to derive equilibrium equations and then impose the uniformity of mass density, then Bernoulli equation (3.23) will act as an additional constraint on the permissible equilibria. However, for uniform mass density, no Bernoulli equation occurs via the variational principle and the computation of the pressure decouples from the PDE problem. Ultimately, the resulting equilibrium equations will be given by (3.19)–(3.22) with $\rho = 1$. This system leads to the equilibrium system of (3.33)–(3.35) with $\rho = 1$, that is, the differential operators on the lhs of (3.33) and (3.34) reduce to the elliptic operator $-\mathcal{L}$ acting on $\mathcal{F}$ and $\mathcal{G}$, respectively. The pressure can be computed from (1.27) upon setting $\partial_t \mathbf{v} = 0$, taking the divergence of the resulting equation and acting with the inverse of the Laplacian operator leading to the following relation

$$p = \Delta^{-1}\nabla \cdot (\mathbf{v} \times \nabla \times \mathbf{v} + \mathbf{J} \times \mathbf{B}^*) - \frac{|\mathbf{v}|^2}{2} - \frac{d_e^2}{2}|\mathbf{J}|^2 \,. \qquad (3.58)$$

If we employ the helically symmetric representation (2.6), (2.7) for the fields $\mathbf{B}^*$, $\mathbf{v}$ and $\mathbf{B}$ and use the equilibrium equations (3.19)-(3.22) with $\rho = 1$, then we can prove that

$$\mathbf{v} \times \nabla \times \mathbf{v} + \mathbf{J} \times \mathbf{B}^* = \nabla \mathcal{M}(\varphi) + \nabla \mathcal{N}(\xi) \,. \qquad (3.59)$$

Therefore, from (3.58) and (3.59), we deduce that the incompressible pressure is given by

$$p = \mathcal{M}(\varphi) + \mathcal{N}(\xi) - \frac{|\mathbf{v}|^2}{2} - \frac{d_e^2}{2}|\mathbf{J}|^2 \,. \qquad (3.60)$$

Below we consider the incompressible Hall MHD equilibrium problem with helical symmetry and derive a special solution describing the so-called Double-Beltrami equilibrium. For an alternative verification of this result obtained upon taking projections of the stationary XMHD equations of motion see Appendix B.

3.3.1 Incompressible Hall MHD equilibria

For incompressible plasmas the HMHD equilibrium system reduces to (3.37)–(3.38) with $\rho = 1$, i.e., we have

$$d_i^2 \mathcal{F}' \mathcal{L} \mathcal{F} \;=\; -k^2(\mathcal{F} + \mathcal{G})\mathcal{F}' - \mathcal{M}' + k^2 \left(d_i^{-2} + 2n\ell k^2 \mathcal{F}'\right)(\varphi - \psi) \,, \qquad (3.61)$$

$$\mathcal{L}\psi \;=\; k^2(\mathcal{F}+\mathcal{G})\mathcal{G}' + \mathcal{N}' + 2n\ell k^4(\mathcal{F}+\mathcal{G}) + k^2(\varphi-\psi)d_i^{-2}\,. \qquad (3.62)$$

The pressure can be computed using (3.60) with $d_e = 0$. To obtain solutions for the fluxes φ and ψ, we need to specify the free functions $\mathcal{F}$, $\mathcal{G}$, $\mathcal{M}$ and $\mathcal{N}$. There exist a particular ansatz for the free functions, for which the system (3.61)–(3.62) permits an analytic solution. In this case the magnetic and velocity fields are superpositions of two Beltrami fields and the functions φ and ψ are expressed as linear combinations of the corresponding poloidal flux functions of the Beltrami fields. The generic linear ansatz, for the system (3.61)–(3.62) is

$$\mathcal{F} = f_0 + f_1\varphi\,, \quad \mathcal{G} = g_0 + g_1\psi\,, \quad \mathcal{M} = m_0 + m_1\varphi\,, \quad \mathcal{N} = n_0 + n_1\psi\,, \qquad (3.63)$$

where f_0, f_1, g_0, g_1, m_0, m_1, n_0, n_1 are constant parameters, leads to the following equations for helically symmetric HMHD equilibria:

$$k^{-2}\mathcal{L}\begin{pmatrix}\varphi\\ \psi\end{pmatrix} = \begin{pmatrix}\mathcal{W}_1 & \mathcal{W}_2\\ \mathcal{W}_3 & \mathcal{W}_4\end{pmatrix}\begin{pmatrix}\varphi\\ \psi\end{pmatrix} + \begin{pmatrix}\mathcal{R}_1\\ \mathcal{R}_2\end{pmatrix}\,, \qquad (3.64)$$

where

$$\begin{aligned}
\mathcal{W}_1 &= \frac{1+2n\ell d_i^2 f_1 k^2}{d_i^4 f_1^2} - \frac{1}{d_i^2}\,,\\
\mathcal{W}_2 &= -\frac{g_1}{d_i^2 f_1} - \frac{1+2n\ell d_i^2 f_1 k^2}{d_i^4 f_1^2}\\
\mathcal{W}_3 &= g_1 f_1 + \frac{1+2n\ell f_1 d_i^2 k^2}{d_i^2}\,,\\
\mathcal{W}_4 &= g_1^2 - \frac{1-2n\ell g_1 d_i^2 k^2}{d_i^2}\,,\\
\mathcal{R}_1 &= -\frac{f_0+g_0}{f_1 d_i^2} - \frac{m_1}{d_i^2 f_1^2 k^2}\,,\\
\mathcal{R}_2 &= g_1(f_0+g_0) + \frac{n_1}{k^2} + 2n\ell k^2(f_0+g_0)\,. \qquad (3.65)
\end{aligned}$$

For $n\,,\ell \neq 0$ we can find a solution to this system assuming $m_1 = n_1 = f_0 = g_0 = 0$:

$$\varphi = \frac{\lambda_+ - g_1}{f_1}\psi_+ + \frac{\lambda_- - g_1}{f_1}\psi_-\,, \qquad \psi = \psi_+ + \psi_-\,, \qquad (3.66)$$

where $\psi_\pm$ are solutions of the equation

$$k^{-2}\mathcal{L}\psi_\pm = \lambda_\pm^2\psi_\pm + 2n\ell\lambda_\pm k^2\psi_\pm\,, \qquad (3.67)$$

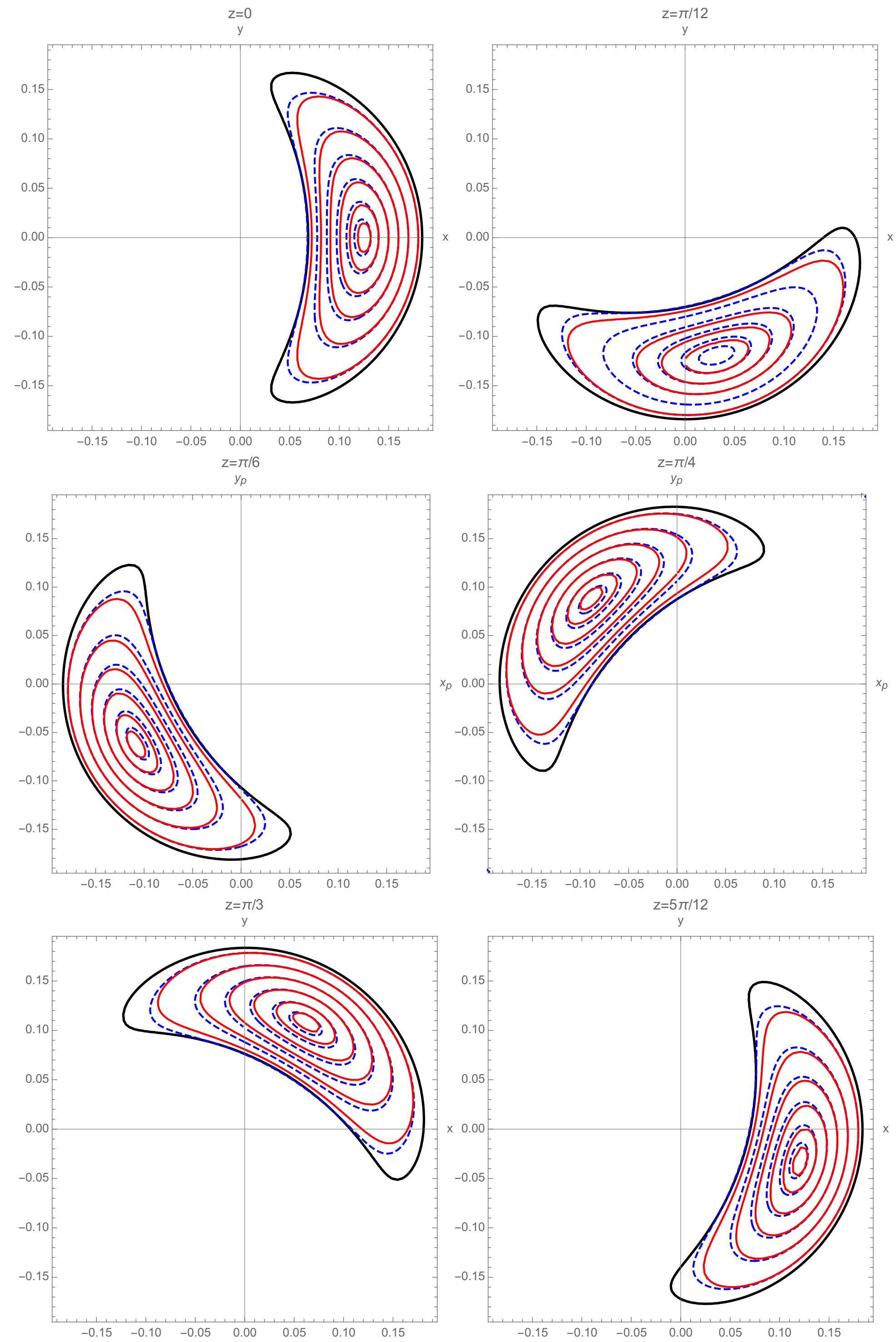


FIGURE 3.9: The magnetic (solid red) and the ion (dashed blue) surfaces of the analytic DB equilibria with helical symmetry in connection with (3.66) and (3.69) in six different sections, namely $z = (0, \pi/12, ..., 5\pi/12)$. The values of the parameters ℓ and n are $\ell = 1$ and $n = 5$ corresponding to five helical windings for distance 2π covered in the z-direction. The contours have been plotted on the (x, y) plane (perpendicular to z-direction).

and the parameters $\lambda_\pm$ are given by

$$\lambda_\pm = \frac{1}{2}\left[\frac{1}{d_i^2 f_1} + g_1 \pm \sqrt{\left(\frac{1}{d_i^2 f_1} + g_1\right)^2 - 4\frac{f_1+g_1}{d_i^2 f_1}}\right] . \tag{3.68}$$

Either solving (3.67) directly or following the construction of [95] (see also [96]) we can obtain the following analytic solutions $\psi_\pm$:

$$\begin{aligned}\psi_\pm \quad &= c_\pm \left[\ell J_0(\lambda_\pm r) - nrJ_1(\lambda_\pm r)\right] \\ &+ \sum_m a_m^\pm \left[\ell\lambda_\pm I_{\ell m}(\sigma_\pm r) + nr\frac{d}{dr}I_{\ell m}(\sigma_\pm r)\right] cos(mu) ,\end{aligned} \tag{3.69}$$

where $\sigma_\pm := \sqrt{m^2n^2 - \lambda_\pm^2}$ and $I_{\ell m}$ denotes the modified Bessel function of the first kind with order ℓm. Parameters $c_\pm$ and $a_m^\pm$ can be specified in connection with the desirable boundary conditions. Functions $\psi_\pm$ are poloidal flux functions of helically symmetric Beltrami fields with Beltrami parameters $\lambda_\pm$. Since the solution is a linear combination of two Beltrami fields and the resulting velocity and magnetic fields satisfy conditions that involve the double curl operator, the resulting solution is called double-Beltrami (DB). Such states, are not only natural solutions of the incompressible Hall MHD equilibrium equations (see [78]) but they occur also as relaxed states via minimization principles [97]. They have been used to construct high-beta equilibria with flows for 1D [78, 98] and axisymmetric systems [99] but not for helically symmetric ones. Here we present a helical DB equilibrium computed by means of (3.66)–(3.69) with $d_i = 0.09$, $f_1 = 4.0$, $g_1 = 2.0$, depicted in figure 3.9. This configuration, possessing closed surfaces, is obtained upon imposing the vanishing of ψ on a set of predetermined points, yielding the values of the free parameters in the truncated series (3.69). We observe that the ion surfaces depart from the electron-magnetic surfaces in a manner similar to the numerical computation of the previous section and also in other studies such as [71] and [90], resulting in a configuration with distinct helical structures for the ions and the electrons.

3.4 Ellipticity condition for the XMHD equilibrium equations

In this section we show how the quasineutrality condition, although it reduces the system of equations that have to be considered for a fully self-consistent description, inserts a peculiarity into the system of equilibrium equations derived in this chapter: the two flux functions representing the electron and the ion contributions are connected through a single Bernoulli equation and a single mass density function. This feature, that is not typical for the complete two-fluid theory, introduces a complication in deriving ellipticity conditions for the XMHD equilibrium system of equations, rendering the condition more involved than those for the two-fluid system. However, there are special cases where the ellipticity condition is reduced to more convenient forms leading to interesting conclusions. Such a case is static equilibria, where we prove that ellipticity is not always the case, despite the fact that if we neglect electron inertia, then the absence of macroscopic flow implies ellipticity, as it is well known in the case of MHD and HMHD.

We are concerned with the problem of ellipticity because the classification of PDEs and systems of PDEs into elliptic, parabolic and hyperbolic ones, is fundamental in the theory of differential equations (e.g. [100]). It is known that boundary value problems (BVPs) with elliptic equations or systems of equations under Dirichlet, Neumann, or Robin boundary conditions are well-posed. For this reason ellipticity is generally desired for equilibrium studies because they rely on solving such boundary value problems. On the other hand hyperbolic equations are usually related to evolutionary problems. It is also known that solutions to elliptic equations have no discontinuous derivatives. Such discontinuities are related to jumps in equilibrium profiles and shock formation, which certainly introduce additional numerical challenges. When describing axisymmetric plasmas within the framework of ordinary MHD, the boundaries between elliptic and hyperbolic regimes are determined by the magnitude of the poloidal flow. Weak poloidal flows render the equilibrium problem elliptic and thus its solution can be attained by standard methods for boundary value problems; however, when poloidal flows have larger magnitudes, then mixed elliptic-hyperbolic regimes, i.e., situations for which the equilibrium system is hyperbolic in one part of the domain and elliptic in the other part, emerge. This implies the existence of discontinuities and jumps in the profiles of quantities such as the plasma density [101]. The connection of strong poloidal sheared flows with the formation of internal transport barriers that are associated with the transition to high confinement modes and whose emergence comes with the formation of steep gradients in equilibrium profiles, establishes a link between mixed elliptic-hyperbolic equilibria with transonic flows and high-mode confinement.

Hence, we understand that it is important to know where the boundaries between elliptic and hyperbolic regimes are located. The ellipticity conditions for single fluid MHD have been derived in several instances e.g. [102, 103, 104]. For the complete two-fluid Grad-Shafranov-Bernoulli equilibrium system, ellipticity conditions are provided in [103], while there are analogous conditions for simplified versions, e.g., in [105] for two-fluid equilibria with massless electrons, in [76, 106, 107] for the Hall MHD model with scalar and anisotropic electron pressure.

For reasons of comparison and completeness we first give below the well-known ellipticity conditions for axisymmetric MHD and HMHD equations and in addition the respective two-fluid conditions. In the context of MHD the axisymmetric Grad-Shafranov-Bernoulli system is elliptic if

$$0 \leq \frac{v_p^2}{v_{Ap}^2} < \frac{c_s^2}{c_s^2 + v_A^2}\,, \quad v_s^2 < v_p^2 < v_A^2\,, \quad v_A^2 < v_p^2 < v_f^2\,, \tag{3.70}$$

where v_p is the poloidal plasma velocity, c_s is the speed of sound, v_A the poloidal Alfvén speed, while v_s and v_f correspond to the slow and fast magnetosonic wave speeds, respectively. We can see that within the framework of ordinary MHD there exist two elliptic regions; the second one, which involves stronger flows, is interrupted by the so-called Alfvén singularity, which occurs when the poloidal flow speed coincides with the poloidal Alfvén speed. This makes the Grad-Shafranov equation singular and a global equilibrium solution cannot be constructed. It is interesting that the speed of sound is not a transition point, the transition points being defined by the trailing cusp speed in the wave-front diagram, $c_s^2/(c_s^2 + v_A^2)$, and the characteristic speeds of the slow and fast magnetosonic waves.

The ellipticity conditions for two-fluid equilibria acquire a much simpler form and only one elliptic region exists, viz

$$v_{ip}^2 < c_{is}^2\,, \quad \text{and} \quad v_{ep}^2 < c_{es}^2\,, \tag{3.71}$$

where $c_{js}^2 = \Gamma p_j/(m_j n_j)$, $j = i, e$ for polytropic gases with adiabatic index Γ, deduced by reversing the hyperbolicity conditions in [103]. In the case of Hall MHD the ellipticity condition, derived in [76], becomes

$$v_p^2 < c_s^2\,, \tag{3.72}$$

where $c_s^2 = c_{is}^2 + c_{es}^2$ holding true for HMHD and XMHD due to the quasineutrality condition.

Conditions (3.71) and (3.72) show hydrodynamic behavior within the two-fluid context, with transitions to hyperbolicity when the poloidal speed reaches the corresponding sound speed. One would expect that since the XMHD model is essentially a quasineutral two-fluid model, would exhibit a similar behavior. However, as we show below, the quasineutrality condition introduces complication in the XMHD formalism. We reveal this complication by deriving the ellipticity condition for the most generic system of axisymmetric XMHD equilibrium equations and later on we discuss some special cases.

Typically, ellipticity is defined for systems of linear PDEs (e.g. for the specific case of second order systems see [108]) because it is a property defined pointwise and is completely depended on the principal symbol of the differential operator; hence, the definition can be extended in order to include quasilinear systems as it is done below.

Consider a second order system of M quasilinear partial differential equations in N independent and M dependent variables of the following form:

$$\sum_{j=1}^{M}\sum_{\ell,n=1}^{N}\tau_{ij}^{\ell n}(x,u,u_x)\frac{\partial^2 u_j}{\partial x_\ell \partial x_n}-f_i(x,u,u_x)=0\,,\qquad i=1,...,M \tag{3.73}$$

where $x=(x_1,...,x_N)\in\mathcal{D}\subset\mathbb{R}^N$, $u=(u_1,...,u_M)\in\mathcal{U}\subset\mathbb{R}^M$, $\tau_{ij}^{\ell n}$ are the coefficients of the second order derivatives in (3.73) and by u_x we denote the first order derivatives of the dependent variables. The classification of the system depends only upon its principal symbol, or characteristic matrix, which for arbitrary, real scalars $\lambda=(\lambda_1,...,\lambda_N)$, is defined as

$$\tau[\lambda]=\left[\sum_{\ell,n=1}^{N}\tau_{ij}^{\ell n}(x,u,u_x)\lambda_\ell\lambda_n\right]\,, \tag{3.74}$$

that is an $M\times M$ matrix with rows and columns labeled by i and j, respectively.

Definition: The second order quasilinear system (3.73) is called elliptic if $\forall x\in\mathcal{D}$, $det\,(\tau[\lambda])\neq 0\ \forall\lambda\neq 0$. That is $det\,(\tau[\lambda])$ has to be positive or negative definite $\forall\lambda\neq 0$.

This definition allows the classification of systems like (3.41)–(3.44), describing axisymmetric barotropic XMHD equilibria. According to this, for the classification of the aforementioned system we are interested in knowing the principal symbol, which depends only on the coefficients of second order derivatives of (3.41)–(3.43). An interesting property of such Grad-Shafranov-Bernoulli (GSB) systems is that the second order derivatives in the flux functions are not only those that appear explicitly in the Grad-Shafranov (GS) equations, but additional terms coming from the involvement of the mass density ρ in the differential operators. This is because partial derivatives of the flux functions are contained in Bernoulli equation due to the presence of the poloidal flow and current density according to the following equations

$$h(\rho)=\mathcal{M}(\varphi)+\mathcal{N}(\xi)-\frac{v_\phi^2}{2}-\frac{v_p^2}{2}-\frac{d_e^2}{2\rho^2}(J_\phi^2+J_p^2)\,, \tag{3.75}$$

$$v_\phi=\frac{1}{r}\frac{\varphi-\xi}{\gamma-\mu}\,,\quad \mathbf{v}_p=\rho^{-1}\nabla\left(\gamma\mathcal{F}+\mu\mathcal{G}\right)\times\nabla\phi\,, \tag{3.76}$$

$$J_\phi=-\frac{\Delta^*\psi}{r}\,,\quad \mathbf{J}_p=\nabla(\mathcal{F}+\mathcal{G})\times\nabla\phi\,, \tag{3.77}$$

therefore, the Bernoulli equation takes the form

$$\begin{aligned}&h(\rho)=\mathcal{M}(\varphi)+\mathcal{N}(\xi)-\frac{1}{2r^2}\frac{(\varphi-\xi)^2}{(\gamma-\mu)^2}-\frac{d_e^2}{2\rho^2r^2}\left(\Delta^*\psi\right)^2\\&-\frac{1}{2\rho^2r^2}\left[(\gamma^2+d_e^2)(\mathcal{F}')^2|\nabla\varphi|^2+(\mu^2+d_e^2)(\mathcal{G}')^2|\nabla\xi|^2\right]\,,\end{aligned} \tag{3.78}$$

where we have used the relation $\gamma\mu = -d_e^2$. It is clear now that $\rho = \rho(r, \varphi, \xi, |\nabla\varphi|^2, |\nabla\xi|^2)$ so $\nabla\rho$ will contain second order derivatives.

To compute the principal symbol of the system let us first expand the differential expressions in the lhs of Eqs. (3.41), (3.42) as follows

$$\begin{aligned}\nabla\cdot\left(\frac{\mathcal{F}'}{\rho}\frac{\nabla\varphi}{r^2}\right) &= \frac{\mathcal{F}'}{\rho r^2}\left(\Delta\varphi - \rho^{-1}\nabla\rho\cdot\nabla\varphi\right) + \text{lower order terms}\,,\\ \nabla\cdot\left(\frac{\mathcal{G}'}{\rho}\frac{\nabla\xi}{r^2}\right) &= \frac{\mathcal{G}'}{\rho r^2}\left(\Delta\xi - \rho^{-1}\nabla\rho\cdot\nabla\xi\right) + \text{lower order terms}\,. \qquad (3.79)\end{aligned}$$

By denoting

$$\rho' := \frac{\partial\rho}{\partial|\nabla\varphi|^2}\,, \qquad \dot\rho := \frac{\partial\rho}{\partial|\nabla\xi|^2}\,, \qquad (3.80)$$

we can write Eqs. (3.79) as

$$\begin{aligned}\nabla\cdot\left(\frac{\mathcal{F}'}{\rho}\frac{\nabla\varphi}{r^2}\right) &= \frac{\mathcal{F}'}{\rho r^2}\left[\Delta\varphi - \rho^{-1}\left(\rho'\nabla|\nabla\varphi|^2 + \dot\rho\nabla|\nabla\xi|^2\right)\cdot\nabla\varphi\right]\\ &\qquad + \text{lower order terms}\,,\\ \nabla\cdot\left(\frac{\mathcal{G}'}{\rho}\frac{\nabla\xi}{r^2}\right) &= \frac{\mathcal{G}'}{\rho r^2}\left(\Delta\xi - \rho^{-1}\left(\rho'\nabla|\nabla\varphi|^2 + \dot\rho\nabla|\nabla\xi|^2\right)\cdot\nabla\xi\right)\\ &\qquad + \text{lower order terms}\,. \qquad (3.81)\end{aligned}$$

After some simple analysis, the XMHD GS equations, i.e. Eqs. (3.81), together with (3.43) can be written as

$$\begin{aligned}&(\gamma^2 + d_e^2)\frac{\mathcal{F}'^2}{\rho r^2}\big[\left(1-\alpha\varphi_r^2\right)\partial_{rr}\varphi \qquad (3.82)\\ &\quad + \left(1-\alpha\varphi_z^2\right)\partial_{zz}\varphi - 2\alpha\varphi_r\varphi_z\partial_{rz}\varphi - \beta\varphi_r\xi_r\partial_{rr}\xi\\ &\quad -\beta\varphi_z\xi_z\partial_{zz}\xi - \beta(\varphi_r\xi_z + \varphi_z\xi_r)\partial_{rz}\xi\big]\\ &\quad + \text{lower order terms} = 0\,,\\ &(\mu^2 + d_e^2)\frac{\mathcal{G}'^2}{\rho r^2}\big[\left(1-\beta\xi_r^2\right)\partial_{rr}\xi \qquad (3.83)\\ &\quad + \left(1-\beta\xi_z^2\right)\partial_{zz}\xi - 2\beta\xi_r\xi_z\partial_{rz}\xi - \alpha\varphi_r\xi_r\partial_{rr}\varphi\\ &\quad -\alpha\varphi_z\xi_z\partial_{zz}\varphi - \alpha(\varphi_r\xi_z + \varphi_z\xi_r)\partial_{rz}\varphi\big]\\ &\quad + \text{lower order terms} = 0\,,\\ &\partial_{rr}\psi + \partial_{zz}\psi + \text{lower order terms} = 0\,, \qquad (3.84)\end{aligned}$$

where $\alpha := 2\rho'/\rho$ and $\beta := 2\dot{\rho}/\rho$. According to definition (3.74), the principal symbol of (3.82)–(3.84) is

$$\tau[\lambda_1, \lambda_2] = \begin{pmatrix} T_{11} & T_{12} & 0 \\ T_{21} & T_{22} & 0 \\ 0 & 0 & \lambda_1^2 + \lambda_2^2 \end{pmatrix}, \tag{3.85}$$

where

$$T_{11} = C_1 \left[(1 - \alpha\varphi_r^2)\lambda_1^2 + (1 - \alpha\varphi_z^2)\lambda_2^2 - 2\alpha\varphi_r\varphi_z\lambda_1\lambda_2\right], \tag{3.86}$$

$$T_{12} = -C_1\beta \left[\varphi_r\xi_r\lambda_1^2 + \varphi_z\xi_z\lambda_2^2 + (\varphi_r\xi_z + \varphi_z\xi_r)\lambda_1\lambda_2\right], \tag{3.87}$$

$$T_{21} = -C_2\alpha \left[\varphi_r\xi_r\lambda_1^2 + \varphi_z\xi_z\lambda_2^2 + (\varphi_r\xi_z + \varphi_z\xi_r)\lambda_1\lambda_2\right], \tag{3.88}$$

$$T_{22} = C_2 \left[(1 - \beta\xi_r^2)\lambda_1^2 + (1 - \beta\xi_z^2)\lambda_2^2 - 2\beta\xi_r\xi_z\lambda_1\lambda_2\right], \tag{3.89}$$

with $C_1 := (\gamma^2 + d_e^2)\mathcal{F}'^2/(\rho r^2)$ and $C_2 := (\mu^2 + d_e^2)\mathcal{G}'^2/(\rho r^2)$. The determinant of the characteristic matrix is

$$\begin{aligned} det(\tau)(\lambda_1, \lambda_2) = C_1C_2(\lambda_1^2 + \lambda_2^2)^2 \big[\lambda_1^2(1 - \alpha\varphi_r^2 - \beta\xi_r^2) + \lambda_2^2(1 - \alpha\varphi_z^2 - \beta\xi_z^2) \\ -2\lambda_1\lambda_2(\alpha\varphi_r\varphi_z + \beta\xi_r\xi_z)\big] =: C_1C_2(\lambda_1^2 + \lambda_2^2)^2 P(\lambda_1, \lambda_2)\,. \end{aligned} \tag{3.90}$$

For free functions $\mathcal{F}(\varphi)$ and $\mathcal{G}(\xi)$ with $\mathcal{F}' \neq 0$ and $\mathcal{G}' \neq 0$ $\forall x \in \mathcal{D}$, the coefficient C_1C_2 can be ignored since it is strictly positive. Clearly for $\mathcal{F}', \mathcal{G}' \neq 0$ and $\lambda_1, \lambda_2 \neq 0$ the determinant can be zero if and only if the homogeneous polynomial $P(\lambda_1, \lambda_2)$ has real roots. Thus the ellipticity condition for XMHD equilibrium equations can be summarized as follows:

$$P(\lambda_1, \lambda_2) \neq 0\,, \quad \forall \lambda_1, \lambda_2 \neq 0\,. \tag{3.91}$$

We can prove, by directly computing the roots of $P(\lambda_1, \lambda_2)$ with respect to either λ_1 or λ_2, that no real roots exist if

$$\begin{aligned} 1 - \alpha|\nabla\varphi|^2 - \beta|\nabla\xi|^2 \\ +\alpha\beta\left(|\nabla\varphi|^2|\nabla\xi|^2 - (\nabla\varphi\cdot\nabla\xi)^2\right) > 0\,. \end{aligned} \tag{3.92}$$

At this point it remains to compute α and β in terms of the equilibrium quantities. This can be done by performing implicit differentiation of equation (3.78) with respect to $|\nabla\varphi|^2$ and $|\nabla\xi|^2$ (e.g. see [102, 76]),

$$\frac{dh}{d\rho}\rho' = -\frac{\gamma^2 + d_e^2}{2\rho^2 r^2}(\mathcal{F}')^2$$

$$+\frac{\rho'}{\rho^3 r^2}\left[(\gamma^2+d_e^2)(\mathcal{F}')^2|\nabla\varphi|^2+(\mu^2+d_e^2)(\mathcal{G}')^2|\nabla\xi|^2\right]\,, \tag{3.93}$$

$$\frac{dh}{d\rho}\dot{\rho}=-\frac{\mu^2+d_e^2}{2\rho^2 r^2}(\mathcal{G}')^2$$
$$+\frac{\dot{\rho}}{\rho^3 r^2}\left[(\gamma^2+d_e^2)(\mathcal{F}')^2|\nabla\varphi|^2+(\mu^2+d_e^2)(\mathcal{G}')^2|\nabla\xi|^2\right]\,, \tag{3.94}$$

and rearranging we find

$$\alpha=-\frac{(\gamma^2+d_e^2)(\mathcal{F}')^2}{\rho^2 r^2\left(c_s^2-\frac{(\gamma^2+d_e^2)(\mathcal{F}')^2|\nabla\varphi|^2+(\mu^2+d_e^2)(\mathcal{G}')^2|\nabla\xi|^2}{\rho^2 r^2}\right)}\,,$$
$$\beta=-\frac{(\mu^2+d_e^2)(\mathcal{G}')^2}{\rho^2 r^2\left(c_s^2-\frac{(\gamma^2+d_e^2)(\mathcal{F}')^2|\nabla\varphi|^2+(\mu^2+d_e^2)(\mathcal{G}')^2|\nabla\xi|^2}{\rho^2 r^2}\right)}\,, \tag{3.95}$$

where $c_s^2 := \rho dh/d\rho = c_{is}^2 + c_{es}^2$ the Alfvén normalized speed of sound. These expressions can be rewritten in terms of physical quantities as follows

$$\alpha = \frac{-(\gamma^2+d_e^2)\mathcal{F}'^2}{\rho^2 r^2\left(c_s^2-v_p^2-\frac{d_e^2}{\rho^2 r^2}|\nabla(rB_\phi)|^2\right)}\,,$$
$$\beta = \frac{-(\mu^2+d_e^2)\mathcal{G}'^2}{\rho^2 r^2\left(c_s^2-v_p^2-\frac{d_e^2}{\rho^2 r^2}|\nabla(rB_\phi)|^2\right)}\,. \tag{3.96}$$

Substituting (3.96) into (3.92) we find

$$\frac{(\gamma^2+d_e^2)(\mu^2+d_e^2)\mathcal{F}'^2\mathcal{G}'^2\left[|\nabla\varphi|^2|\nabla\xi|^2-(\nabla\varphi\cdot\nabla\xi)^2\right]}{\rho^4 r^4\left(v_p^2+\frac{d_e^2}{\rho^2 r^2}|\nabla(rB_\phi)|^2-c_s^2\right)^2}$$
$$+\frac{1}{1-\left(v_p^2+\frac{d_e^2}{\rho^2 r^2}|\nabla(rB_\phi)|^2\right)/c_s^2}>0\,. \tag{3.97}$$

This is the ellipticity condition for the complete system of axisymmetric XMHD equilibrium equations. We observe that since the first term is always non-negative a sufficient (but not necessary) condition for ellipticity is

$$v_p^2+\frac{d_e^2}{\rho^2 r^2}|\nabla(rB_\phi)|^2<c_s^2\,. \tag{3.98}$$

Observe in (3.97) that setting $d_e = 0$, i.e. neglecting electron inertia, we recover the Hall MHD ellipticity condition $v_p^2 < c_s^2$. Now let us assume that our equations describe macroscopically static equilibria i.e. $\mathbf{v} \equiv 0$. Then from the expression for v_ϕ in (3.76), we conclude that $\varphi = \xi$. Thus (3.97) reduces to

$$\frac{d_e^2}{\rho^2 r^2}|\nabla(rB_\phi)|^2<c_s^2\,. \tag{3.99}$$

Therefore, in principle, elliptic-hyperbolic transitions are possible even for zero macroscopic flow, something that cannot happen within the framework of the MHD and HMHD. This is indeed plausible because static XMHD equilibrium does not imply that the ion and electron fluids are strictly static as well – if that were the case, there would be no current at all, and in addition electron mass is not neglected. However, we need to clarify that the violation of the ellipticity condition (3.99) would require rather peculiar conditions, i.e. very high current density, since $|\nabla(rB_\phi)|^2/r^2$ is the poloidal current density squared and very low mass density, because the speed of sound decreases, if for example a polytropic equation of state is adopted ($p \propto \rho^\Gamma$), while the lhs of (3.99) increases with density decrease. Therefore, a transition to the hyperbolic regime requires sufficiently small mass density or/and sufficiently high poloidal current density.

In addition, we point out that (3.99) holds also for purely toroidal flows ($v_p = 0$), because in that case $\varphi = f(\xi)$ (see Eq. (3.36)), so again the first term of (3.97) vanishes. Another case that admits a simplified version of the ellipticity condition (3.97) is when one of the two free functions $\mathcal{F}$, $\mathcal{G}$ is constant, say $\mathcal{G}' = 0$. In this case poloidal flow is present and the flow surfaces coincide with the level sets of the stream function φ. For $\mathcal{G}' = 0$, Eq. (3.97) reduces to Eq. (3.98) that represents now both a necessary and sufficient ellipticity condition.

As a final point we address the following reasonable question: why does the more generic case of two-fluid equilibria possesses an ellipticity condition simpler in form? As stated before, the quasineutrality condition combined with the finite electron inertia (which is absent in the quasineutral HMHD model) is the source of the complication, for it causes the two stream functions to be related through a single Bernoulli equation. In the two-fluid case there exist two Bernoulli equations for the two mass densities, each one of which contains a dependence on the gradient of the corresponding stream function and each GS equation contains only the corresponding mass density function. As a consequence the principal symbol has only diagonal elements and the ellipticity condition for each fluid becomes trivial because it results from the requirement that the diagonal elements must have no real roots. This requirement leads eventually to the pair of inequalities (3.71) instead of the single inequality (3.97).

Chapter 4

Stability Analysis of extended MHD equilibria

In this chapter we construct and examine three different functionals, representing the energy of different kinds of perturbations, in order to derive sufficient stability conditions for generalized MHD models. First, $\delta^2\mathcal{H}_C[\boldsymbol{u}_e, \delta\boldsymbol{u}]$ and $\delta^2\mathcal{H}_{da}[\boldsymbol{u}_e, \delta\boldsymbol{u}_{da}]$, which represent the second variation of the EC functional and the energy of the DA perturbations, respectively, are constructed. Both are expressed in Eulerian variables (the subscript e denoting equilibrium). The third functional, $\delta^2\mathcal{H}_{la}[\mathbf{q}_e, \boldsymbol{\pi}_e, \mathbf{A}_e, \Phi_e; \delta\mathbf{q}, \delta\boldsymbol{\pi}, \delta\mathbf{A}, \delta\Phi]$, is the second order variation of the Hamiltonian within a mixed Eulerian-Lagrangian description, where the fluid variables are described in the Lagrangian picture while the electromagnetic field in the Eulerian picture. The results of this chapter can be found in [109].

The chapter is organized as follows: Section 4.1 is devoted to a brief comparison between the three approaches. In Section 4.2 we employ the energy-Casimir method for studying the stability of axisymmetric XMHD equilibria by computing the second order variation of the EC functional. Several sufficient stability criteria are derived, concerning either special equilibria or special perturbations. In this way we fulfill the four step procedure described in Section 1.3.3. In Section 4.3 we find the dynamically accessible variations for the XMHD model. In addition the second order, DAV of the Hamiltonian is computed and used to establish a stability criterion for generic equilibria. Finally, in Section 4.4 we compute the second order variation of the Lagrangian in a mixed Eulerian-Lagrangian framework and furthermore we employ a Lagrange-Euler map to express the Lagrangian completely in terms Eulerian coordinates. These results are used to construct the Hamiltonians for the linearized dynamics of the quasineutral two-fluid model and for the Hall MHD model in Section 4.5.

4.1 Comparison of stability methods

Although the basics and the underlying theory of these three methods have been described in Chapter 1 we briefly delineate in this section and also throughout the whole chapter some characteristic features of each method.

In general, Lagrangian stability, being applicable for all possible equilibria and also considering perturbations that are not dynamically restricted or constrained by spatial symmetry, appears to be in practice the most generic method. Lagrangian variations are expressed in terms of Lagrangian displacement vectors $\boldsymbol{\zeta}$ which simply represent the displacement of a perturbed fluid trajectory with respect to its stationary counterpart, no matter what is the mechanism or the cause of this displacement. However, the variations of the fields are generated through certain relations involving the Lagrangian displacements. Thus from a dynamical point of view, they are not completely arbitrary like the EC variations.

The EC method inserts certain restrictions because its applicability is not always guaranteed, since it requires a sufficient number of Casimir invariants in order to be established. This is the reason why in 3D systems EC stability is usually not feasible other than special cases when there exist some kind of Ertel's invariants providing additional Casimirs [10]. If a continuous spatial symmetry is present, the usual helicity Casimirs are converted to infinite families of invariants in view of the symmetric decomposition of the fields, thus rendering the EC method applicable, as for example in [49, 85, 87, 110] for the MHD model. This decomposition restricts the variations to respect the system's geometric symmetry as well. Also EC method is applicable only for assessing the stability of EC equilibria, which do not contain all possible steady states. However, despite these setbacks the EC method exhibits some advantages: it is easy to implement, especially when the study of equilibrium has been carried out within the EC framework, like in the present study, it is easier to prove the positive definiteness of the corresponding Lyapunov functionals, and the dynamics of the variations is arbitrary.

As mentioned in Chapter 1, within the noncanonical Hamiltonian framework one can consider also the so-called dynamically accessible variations (DAVs) (e.g. [8, 12, 111]). Recall that the main advantages of this approach are, that it is valid for generic equilibria and not only for EC ones and also it allows three dimensional perturbations. On the other hand, dynamical accessibility restricts the perturbed trajectories onto the symplectic leaves, which are essentially the level sets of the Casimirs. Note however that DAVs are perhaps the most probable kind of perturbations when no external interventions or dissipative processes that violate the ideal dynamics and the closedness of the system take place. This is because, in the absence of such agents, any perturbed state should be accessible by the ideal dynamics of the model under consideration, which preserve the Casimirs. If perturbations away from symplectic leaves occur, these must come from physics outside the dynamical model. Viewed this way, DAV stability is quite natural to consider.

In terms of dynamics only, if all three methods are applied under the same conditions, then the EC can be regarded as the most general and DA as the most restricted one.

4.2 Energy-Casimir stability of axisymmetric equilibria

4.2.1 Axisymmetric XMHD energy-Casimir functional

In the previous chapter the equilibrium equations for helically symmetric and axisymmetric barotropic plasmas described by XMHD, were derived upon using the energy-Casimir principle. Here, we compute the second order variation of the EC functional restricting our analysis to the axisymmetric case. From (2.6)–(2.7), setting $\ell = 0$ and $n = -1$, one can see that the axisymmetric velocity and magnetic fields can be Helmholtz-decomposed as follows

$$\mathbf{v} = rv_\phi \nabla\phi + \nabla\chi \times \nabla\phi + \nabla\Upsilon\,, \tag{4.1}$$

$$\mathbf{B} = rB_\phi \nabla\phi + \nabla\psi \times \nabla\phi\,, \tag{4.2}$$

inducing a similar form for the generalized magnetic field $\mathbf{B}^*$. From Eqs. (2.86)–(2.89) with $\ell = 0$ we can easily obtain the following axisymmetric Casimirs

$$\mathcal{C}_1 = \int_D d^2x\,(r^{-1}B^*_\phi + \gamma\Omega)\mathcal{F}(\psi^* + \gamma r v_\phi)\,, \tag{4.3}$$

$$\mathcal{C}_2 = \int_D d^2x\,(r^{-1}B^*_\phi + \mu\Omega)\mathcal{G}(\psi^* + \mu r v_\phi)\,, \tag{4.4}$$

$$\mathcal{C}_3 = \int_D d^2x\,\rho\mathcal{M}(\psi^* + \gamma r v_\phi)\,, \tag{4.5}$$

$$\mathcal{C}_4 = \int_D d^2x\,\rho\mathcal{N}(\psi^* + \mu r v_\phi)\,, \tag{4.6}$$

(4.7)

where $\Omega := (\nabla \times \mathbf{v}_\perp)\cdot\nabla\phi$ with $\mathbf{v}_\perp := \nabla\chi \times \nabla\phi + \nabla\Upsilon$ and

$$\psi^* = \psi - d_e^2\rho^{-1}\Delta^*\psi\,, \tag{4.8}$$

$$B^*_\phi = B_\phi - d_e^2 r\nabla\cdot\left[r^{-2}\rho^{-1}\nabla(rB_\phi)\right]\,, \tag{4.9}$$

which are the axisymmetric limits of (2.36) and (2.35), respectively. Also from (2.34) one can find that the axisymmetric Hamiltonian is given by

$$\mathcal{H} = \int_D d^2x\left(\rho\frac{v_\phi^2}{2} + \rho\frac{|\nabla\chi|^2}{2r^2} + \rho\frac{|\nabla\Upsilon|^2}{2}\right.$$

$$
+\rho[\Upsilon,\chi]+\rho U(\rho)+\frac{B_\phi^* B_\phi}{2}+\frac{\nabla\psi^*\cdot\nabla\psi}{2r^2}\Bigg)
$$
$$
=\int_D d^2x\left(\rho\frac{v_\phi^2}{2}+\rho\frac{|\mathbf{v}_\perp|^2}{2}+\rho U(\rho)+\frac{B_\phi^* B_\phi}{2}+\frac{\nabla\psi^*\cdot\nabla\psi}{2r^2}\right). \tag{4.10}
$$

Requiring the vanishing of $\delta\mathcal{H}_C=\delta(\mathcal{H}-\sum_i\mathcal{C}_i)$, yields the EC equilibrium equations, given by (3.17)–(3.22) with $\ell=0, n=-1$ therein, which can be written in the Grad-Shafranov-Bernoulli form (3.41)–(3.44). To proceed with stability analysis it is convenient having the first-order variation of the axisymmetric Hamiltonian written down as

$$
\delta\mathcal{H}_C=\int_D d^2x\Bigg\{\Bigg[h(\rho)-\mathcal{M}-\mathcal{N}+\frac{v_\phi^2}{2}+\frac{|\mathbf{v}_\perp|^2}{2}
$$
$$
+\frac{d_e^2}{2r^2\rho^2}\left((\Delta^*\psi)^2+|\nabla(rB_\phi)|^2\right)\Bigg]\delta\rho+\left[B_\phi-r^{-1}(\mathcal{F}+\mathcal{G})\right]\delta B_\phi^*
$$
$$
+\left[\rho v_\phi-\gamma r(r^{-1}B_\phi^*+\gamma\Omega)\mathcal{F}'-\mu r(r^{-1}B_\phi^*+\mu\Omega)\mathcal{G}'-\gamma r\rho\mathcal{M}'-\mu r\rho\mathcal{N}'\right]\delta v_\phi
$$
$$
-\left[r^{-2}\Delta^*\psi+(r^{-1}B_\phi^*+\gamma\Omega)\mathcal{F}'+(r^{-1}B_\phi^*+\mu\Omega)\mathcal{G}'+\rho\mathcal{M}'+\rho\mathcal{N}'\right]\delta\psi^*
$$
$$
+\left[\rho\mathbf{v}_\perp-\gamma\nabla\mathcal{F}\times\nabla\phi-\mu\nabla\mathcal{G}\times\nabla\phi\right]\cdot\delta\mathbf{v}_\perp\Bigg\}, \tag{4.11}
$$

which will be used for a straightforward computation of the second order variation.

4.2.2 Second order variation

The expressions into the square brackets in (4.11) vanish on the EC equilibrium solution, therefore the second order variation would involve only first order variations of the field variables. After some simple manipulations we can show that $\delta^2\mathcal{H}_C[\boldsymbol{u}_e,\delta\boldsymbol{u}]$ can be written in the following form:

$$
\delta^2\mathcal{H}_C[\boldsymbol{u}_e;\delta\boldsymbol{u}]=\int_D d^2x\Bigg\{\frac{d_e^2}{\rho r^2}|\nabla(r\delta B_\phi)|^2+\frac{|\nabla\delta\psi|^2}{r^2}
$$
$$
+\frac{d_e^2r^2}{\rho}\left[\nabla\cdot\left(r^{-2}\nabla\delta\psi\right)\right]^2+\rho\left(\delta v_\phi+\rho^{-1}v_\phi\delta\rho\right)^2
$$
$$
+\rho\left|\delta\mathbf{v}_\perp+\rho^{-1}\mathbf{v}_\perp\delta\rho\right|^2-2\frac{d_e^2}{r^2\rho}\nabla(\delta\mathcal{F}+\delta\mathcal{G})\cdot\nabla(r\delta B_\phi)
$$
$$
+2\frac{d_e^2}{r^2\rho^2}\nabla(\delta\mathcal{F}+\delta\mathcal{G})\cdot\nabla(rB_\phi)\delta\rho
$$
$$
-2[(\gamma\nabla\delta\mathcal{F}+\mu\nabla\delta\mathcal{G})\times\nabla\phi]\cdot\delta\mathbf{v}_\perp\Bigg\}+\mathcal{Q}, \tag{4.12}
$$

where

$$
\mathcal{Q}=\int_D d^2x\,(\delta B_\phi\;\delta\varphi\;\delta\xi\;\delta\rho)\,\mathcal{A}\,(\delta B_\phi\;\delta\varphi\;\delta\xi\;\delta\rho)^T, \tag{4.13}
$$

with

$$\mathcal{A} = \begin{pmatrix} 1 & A_{\varphi B_\phi} & A_{\xi B_\phi} & 0 \\ A_{\varphi B_\phi} & A_{\varphi\varphi} & 0 & A_{\varphi\rho} \\ A_{\xi B_\phi} & 0 & A_{\xi\xi} & A_{\xi\rho} \\ 0 & A_{\varphi\rho} & A_{\xi\rho} & A_{\rho\rho} \end{pmatrix} . \tag{4.14}$$

The elements of $\mathcal{A}$ are given explicitly by

$$A_{\varphi\varphi} = -\left(r^{-1}B_\phi^* + \gamma\Omega\right)\mathcal{F}'' - \rho\mathcal{M}'' \tag{4.15}$$

$$A_{\xi\xi} = -\left(r^{-1}B_\phi^* + \mu\Omega\right)\mathcal{G}'' - \rho\mathcal{N}'' \tag{4.16}$$

$$A_{\varphi B_\phi} = -r^{-1}\mathcal{F}' , \quad A_{\xi B_\phi} = -r^{-1}\mathcal{G}' , \tag{4.17}$$

$$A_{\varphi\rho} = -\mathcal{M}' , \quad A_{\xi\rho} = -\mathcal{N}' \tag{4.18}$$

$$A_{\rho\rho} = \rho^{-1}\Bigg[c_s^2 - v_\phi^2 - |\mathbf{v}_\perp|^2 - \frac{d_e^2}{\rho^2}\left(r^2\left[\nabla\cdot\left(r^{-2}\nabla\psi\right)\right]^2 + r^{-2}\left|\nabla(rB_\phi)\right|^2\right)\Bigg] , \tag{4.19}$$

where $c_s^2 := \rho h'(\rho)$. In deriving (4.12) we performed integrations by parts, omitted the surface integrals and completed squares in terms involving the mass density and velocity field variations. Also we used the following expressions for $\delta\psi^*$ and δB_ϕ^*

$$\delta\psi^* = \delta\psi - d_e^2\frac{\Delta^*\delta\psi}{\rho} + d_e^2\frac{\Delta^*\psi}{\rho^2}\delta\rho , \tag{4.20}$$

$$\delta B_\phi^* = \delta B_\phi - d_e^2 r\nabla\cdot\left[\frac{\nabla(r\delta B_\phi)}{r^2\rho}\right] + d_e^2 r\nabla\cdot\left[\frac{\nabla(rB_\phi)}{r^2\rho^2}\delta\rho\right] , \tag{4.21}$$

which can easily deduced from Eqs. (4.8) and (4.9), respectively.

For $\mathcal{Q}$ alone to be positive definite, matrix $\mathcal{A}$ has to be positive definite as well, which is equivalent to the requirement that the principal minors of $\mathcal{A}$ satisfy

$$A_{\varphi\varphi} - A_{\varphi B_\phi}^2 > 0 , \tag{4.22}$$

$$A_{\xi\xi}(A_{\varphi\varphi} - A_{\varphi B_\phi}^2) - A_{\varphi\varphi}A_{\xi B_\phi}^2 > 0 , \tag{4.23}$$

$$A_{\rho\rho}\left[A_{\xi\xi}(A_{\varphi\varphi} - A_{\varphi B_\phi}^2) - A_{\varphi\varphi}A_{\xi B_\phi}^2\right] + (A_{\varphi B_\phi}A_{\xi\rho} - A_{\xi B_\phi}A_{\varphi\rho})^2 - A_{\xi\xi}A_{\varphi\rho}^2 - A_{\varphi\varphi}A_{\xi\rho}^2 > 0 . \tag{4.24}$$

For conditions (4.22)–(4.23) to hold, it is necessary that $A_{\varphi\varphi} > 0$ and $A_{\xi\xi} > 0$. However, $\mathcal{Q} > 0$ does not imply stability because there are several indefinite terms in $\delta^2\mathcal{H}_C$. More precisely, the first five terms in $\delta^2\mathcal{H}_C$ are always non-negative, with the magnetic terms expressing the magnetic field line bending while the remaining two terms contain kinetic energy and compressional contributions of the perturbation. These kinetic-compressional terms constitute an example of the typical non-separability between the kinetic and potential energies in systems with macroscopic flows, rendering the resulting stability conditions sufficient but not necessary (see 1). The non-separability is even more severe, since kinetic and potential energy contributions are intertwined also via additional terms in $\delta^2\mathcal{H}_C$ reflecting the fact that in the two fluid framework the coupling between flows and magnetic fields is more complicated. In particular, what really makes life difficult, are the last three terms into the curly bracket in (4.12) because they are clearly sources of indefiniteness, a characteristic which has been detected also in previous energy-Casimir stability analyses of similar models e.g. [84, 112], and can potentially be related to linear instability or the presence of Negative Energy Modes (NEMs). Both can lead to disastrous destabilization and loss of confinement. In order to remove the indefiniteness, we can eliminate or conflate these "problematic" terms into others in view of certain constraints imposed on the variations δB_ϕ^* and $\delta\Omega$ or by considering special equilibria.

4.2.3 Special equilibria

Extended MHD

For purely toroidal flow and current i.e. $\mathcal{F}' = \mathcal{G}' = 0$, it is clear that $\mathcal{Q} > 0$ implies $\delta^2\mathcal{H}_C > 0$. For our special class of equilibria, we have $A_{\varphi B_\phi} = A_{\xi B_\phi} = 0$ and consequently conditions (4.22)–(4.24) yield

$$\mathcal{M}'' < 0\,, \quad \mathcal{N}'' < 0\,, \tag{4.25}$$

$$\mathcal{M}''\mathcal{N}''\left[c_s^2 - v_\phi^2 - \frac{d_e^2}{\rho^2}\left(r^2\left[\nabla\cdot\left(r^{-2}\nabla\psi\right)\right]^2\right)\right] + \mathcal{M}''(\mathcal{N}')^2 + \mathcal{N}''(\mathcal{M}')^2 > 0\,. \tag{4.26}$$

The first two conditions imply that $\mathcal{M}$ and $\mathcal{N}$, have to be concave functions, then for condition (4.26) to be satisfied, the quantity inside the square bracket must be necessarily positive, that is the toroidal velocity, modified by an electron inertial correction, has to be lower than the speed of sound, thus preventing shock formation.

Hall MHD

In the limit $d_e \to 0$, $\mu \to 0$ as well and there is only one indefinite term in (4.12) which can be removed upon selecting $\mathcal{F}' = 0$. In this case the flow is purely toroidal, but there is poloidal current created by the electrons. From (4.22)–(4.24) we obtain the following sufficient stability conditions

$$\mathcal{M}'' < 0\,, \tag{4.27}$$

$$r^{-2}\mathcal{G}\mathcal{G}'' + \rho\mathcal{N}'' + r^{-2}(\mathcal{G}')^2 < 0\,, \tag{4.28}$$

$$\left[\mathcal{M}''(c_s^2 - v_\phi^2) + (\mathcal{M}')^2\right]\left[r^{-2}\mathcal{G}\mathcal{G}'' + \rho\mathcal{N}'' + r^{-2}(\mathcal{G}')^2\right] + \rho\mathcal{M}''(\mathcal{N}')^2 > 0\,. \tag{4.29}$$

The conditions above necessarily entail $c_s^2 - v_\phi^2 > 0$. This special case is interesting because the stability condition is expressed explicitly in terms of equilibrium quantities and, furthermore, it allows us to study the stability of nontrivial equilibria. For this reason we proceed by constructing a Hall MHD equilibrium with purely toroidal rotation and applying the criterion (4.27)–(4.29). From $\delta\mathcal{H}_c = 0$ (see (4.11)), setting $d_e = 0$ and imposing $\mathbf{v}_\perp = \delta\mathbf{v}_\perp = 0$ we can easily extract the equilibrium equations of interest. These are

$$\Delta^*\psi + \mathcal{G}\mathcal{G}'(\psi) + \rho\frac{\varphi-\psi}{d_i^2} + r^2\rho\mathcal{N}'(\psi) = 0\,, \tag{4.30}$$

$$h(\rho) = \mathcal{M}(\varphi) + \mathcal{N}(\psi) - \frac{v_\phi^2}{2}\,, \tag{4.31}$$

$$B_\phi = r^{-1}\mathcal{G}(\psi)\,,\ v_\phi = d_i r\mathcal{M}'(\varphi)\,, \tag{4.32}$$

$$\varphi - d_i^2 r^2\mathcal{M}'(\varphi) = \psi\,, \tag{4.33}$$

where we have used the definition of φ to write $v_\phi = \frac{\varphi-\psi}{d_i r}$. Additionally, we consider the following nonlinear ansatz for the free functions $\mathcal{G}$, $\mathcal{M}$ and $\mathcal{N}$

$$\begin{aligned}
\mathcal{G} =&\quad g_0 + g_1\psi + \frac{1}{2}g_2\psi^2 + \frac{1}{3}g_3\psi^3\,,\\
\mathcal{M} =&\quad m_0 + m_1\varphi + \frac{1}{2}m_2\varphi^2 + \frac{1}{3}m_3\varphi^3\,,\\
\mathcal{N} =&\quad n_0 + n_1\psi + \frac{1}{2}n_2\psi^2 + \frac{1}{3}n_3\psi^3\,,
\end{aligned} \tag{4.34}$$

in which case Eq. (4.33) assumes the following solutions

$$\varphi_\pm = \frac{1 - d_i^2 m_2 r^2 \pm \sqrt{(1-d_i^2 m_2 r^2)^2 - 4d_i^2 m_3 r^2(\psi + d_i^2 m_1 r^2)}}{2d_i^2 m_3 r^2}\,, \tag{4.35}$$

and we choose $m_1 = 0$. This implies that there exists a solution for which the stream function φ vanishes wherever $\psi = 0$, therefore ψ and φ can satisfy the same boundary condition. We consider an adiabatic equation of state i.e. $h(\rho) = \Gamma/(\Gamma-1)p_1\rho^{\Gamma-1}$, where $\Gamma = 5/3$ is the adiabatic index and p_1 is a constant. Then Eq. (4.30) was solved numerically using a modification of the solver that was created for the computation of the numerical equilibrium in the previous chapter. The computational domain is the same up-down poloidally asymmetric domain with prescribed diverted boundary having a lower x-point. It is not difficult to adjust the free parameters in (4.34) to make conditions (4.27)–(4.28) be satisfied everywhere in the plasma. However, when it comes

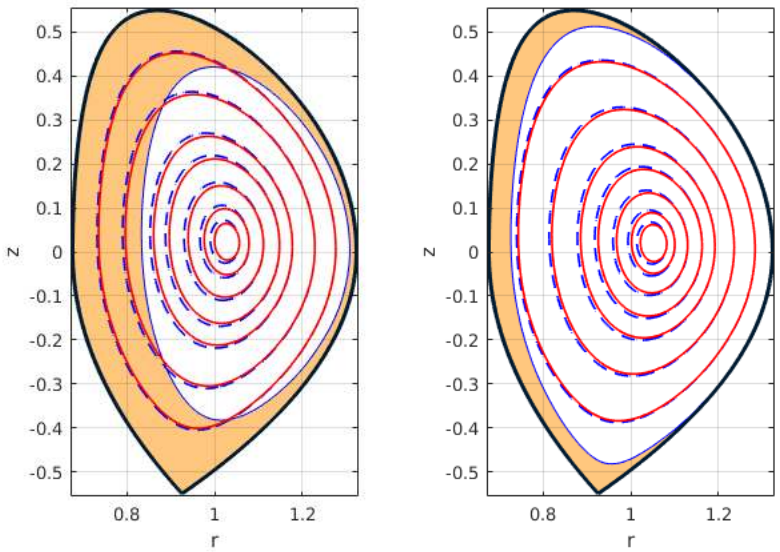


FIGURE 4.1: Stability diagrams for two ITER-like equilibria with maximum $\beta \sim 2\%$ (left) and $\sim 20\%$ (right). In the coloured regions all three conditions (4.27)–(4.29) are satisfied. The Hall parameter is $d_i = 0.04$ in both cases. Solid red lines represent the magnetic surfaces ($\psi = const.$), while the dashed blue ones are surfaces of constant angular velocity ($\varphi = const.$) (see Eq. (4.32)).

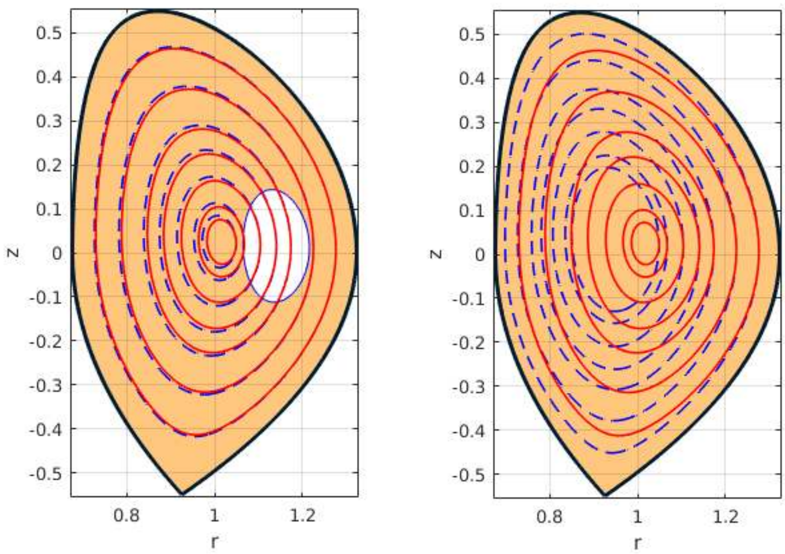


FIGURE 4.2: Stability diagrams for equilibria with maximum $\beta \sim 0.8\%$ with $d_i = 0.04$ (left) and $d_i = 0.24$ (right). While for $d_i = 0.04$ there is a hole within which (4.29) is not satisfied, increasing d_i results in a completely stable configuration, under EC variations.

to (4.29) we observe that for beta values relevant to Tokamak plasmas, i.e. $\beta > 1\%$, the condition is satisfied only within a narrow annular region, wider on the high field side and narrower on the low field side. For $\beta > 10\%$ this region is even narrower forming a thin layer spreading across the high field side (Fig. 4.1). We were able to find equilibria that satisfy all three conditions (4.27)–(4.29) all over the computational domain, only for $\beta < 1\%$. This indicates that condition (4.29) is potentially related with the stabilization of pressure driven modes. To capture the influence of the Hall parameter d_i on stability, we considered an equilibrium with $d_i = 0.04$ where all three stability conditions are satisfied everywhere outside a small region near the core. Then we increased gradually d_i observing that this region was continuously shrinking until it disappeared for $d_i = 0.24$. Thereby, we conclude that increasing d_i the stability properties may be improved (see Fig. 4.2). We also corroborated that if we include the linear term in $\mathcal{M}$, related to rigid rotation and therefore being intrinsically destabilizing, shrinks the "stable" region towards the high field side. In closing we underline that an equilibrium that fails to satisfy the stability conditions is not necessarily unstable since the criteria we derived are only sufficient.

4.2.4 Conditional stability (constrained variations)

As mentioned earlier, the indefiniteness in $\delta^2\mathcal{H}_C$ comes from the terms in (4.12) containing $\delta\mathcal{F}$ and $\delta\mathcal{G}$ and multiplied by $\nabla\times\delta\mathbf{v}_\perp$, δB_ϕ and $\delta\rho$. Hence, a simple way to get rid of the indefiniteness is to assume $\delta\rho = \delta B_\phi = \nabla\times\delta\mathbf{v}_\perp = 0$. However, such a severe restriction of the permitted perturbations should be justified on physical grounds. Another possibility is to eliminate or conflate these terms by other means. A way to do so is to partially minimize the functional (4.12) with respect to $\delta\mathbf{v}_\perp$ and δB_ϕ. This is a standard procedure to obtain simplified forms of the Lyapunov functional and improved stability criteria e.g. see [113, 114, 13, 49]. The minimization can be realized upon considering $\delta^2\mathcal{H}_C$ as a functional of the variations $\delta\mathbf{u}$ and set its variation with respect to δB_ϕ and $\delta\mathbf{v}_\perp$ equal to zero. The resulting Euler-Lagrange equations

$$\delta B_\phi = r^{-1}(\delta\mathcal{F}+\delta\mathcal{G})\,, \tag{4.36}$$

$$\delta\mathbf{v}_\perp = -\mathbf{v}_\perp\delta\rho + (\gamma\nabla\delta\mathcal{F}+\mu\nabla\delta\mathcal{G})\times\nabla\phi\,, \tag{4.37}$$

are indeed minimizers of the functional, since the second variation with respect to δB_ϕ and $\delta\mathbf{v}_\perp$ is positive definite. Upon substituting Eqs. (4.36)–(4.37) into (4.12) we find

$$\delta^2\tilde{\mathcal{H}}_C = \int d^2x\left\{\frac{|\nabla\delta\psi|^2}{r^2} + \frac{d_e^2 r^2}{\rho}\left[\nabla\cdot\left(r^{-2}\nabla\delta\psi\right)\right]^2 + \rho\left(\delta v_\phi + \rho^{-1}v_\phi\delta\rho\right)^2\right.$$
$$+\left[h'(\rho) - \frac{|\mathbf{v}|^2}{\rho} - \frac{d_e^2}{\rho^3}|\mathbf{J}|^2\right](\delta\rho)^2 - 2\mathcal{M}'(\varphi)\delta\rho\delta\varphi - 2\mathcal{N}'(\xi)\delta\rho\delta\xi$$

$$
\begin{aligned}
& - \left[(r^{-1}B^*_\phi + \gamma\Omega)\mathcal{F}''(\varphi) + \rho\mathcal{M}''(\varphi)\right](\delta\varphi)^2 \\
& - \left[(r^{-1}B^*_\phi + \mu\Omega)\mathcal{G}''(\xi) + \rho\mathcal{N}''(\xi)\right](\delta\xi)^2 \\
& -r^{-2}(\delta\mathcal{F}+\delta\mathcal{G})^2 - \frac{d_e^2}{r^2\rho}|\nabla\delta\mathcal{F}+\nabla\delta\mathcal{G}|^2 - \frac{|\gamma\nabla\delta\mathcal{F}+\mu\nabla\delta\mathcal{G}|^2}{r^2\rho} \\
& +\frac{2}{\rho}\mathbf{v}_\perp\cdot\left[(\gamma\nabla\delta\mathcal{F}+\mu\nabla\delta\mathcal{G})\times\nabla\phi\right]\delta\rho + \frac{2d_e^2}{r^2\rho^2}\nabla(rB_\phi)\cdot\nabla(\delta\mathcal{F}+\delta\mathcal{G})\delta\rho\Bigg\}. \qquad (4.38)
\end{aligned}
$$

Using the following equations

$$
\mathbf{v}_\perp = \rho^{-1}\nabla(\gamma\mathcal{F}+\mu\mathcal{G})\times\nabla\phi\,, \qquad (4.39)
$$

$$
B_\phi = \frac{\mathcal{F}+\mathcal{G}}{r}\,, \qquad (4.40)
$$

which can be deduced from (4.11), we find

$$
\begin{aligned}
\delta^2\tilde{\mathcal{H}}_C = & \int d^2x\left\{\frac{|\nabla\delta\psi|^2}{r^2} + \frac{d_e^2r^2}{\rho}\left[\nabla\cdot\left(r^{-2}\nabla\delta\psi\right)\right]^2 + \rho\left(\delta v_\phi + \rho^{-1}v_\phi\delta\rho\right)^2\right. \\
& -\left[r^{-2}(\mathcal{F}')^2 + \frac{\gamma^2+d_e^2}{\rho r^2}|\nabla\mathcal{F}'|^2\right](\delta\varphi)^2 - \left[r^{-2}(\mathcal{G}')^2 + \frac{\mu^2+d_e^2}{\rho r^2}|\nabla\mathcal{G}'|^2\right](\delta\xi)^2 \\
& -2\frac{\mathcal{F}'\mathcal{G}'}{r^2}\delta\varphi\delta\xi + 2\frac{\gamma^2+d_e^2}{\rho^2r^2}\mathcal{F}'\mathcal{F}''|\nabla\varphi|^2\delta\rho\delta\varphi + 2\frac{\mu^2+d_e^2}{\rho^2r^2}\mathcal{G}'\mathcal{G}''|\nabla\xi|^2\delta\rho\delta\xi \\
& +2\frac{\gamma^2+d_e^2}{\rho^2r^2}(\mathcal{F}')^2\delta\rho\nabla\varphi\cdot\nabla\delta\varphi + 2\frac{\mu^2+d_e^2}{\rho^2r^2}(\mathcal{G}')^2\delta\rho\nabla\xi\cdot\nabla\delta\xi \\
& -\frac{\gamma^2+d_e^2}{\rho r^2}\left[(\mathcal{F}')^2|\nabla\delta\varphi|^2 + \delta\varphi\nabla(\mathcal{F}')^2\cdot\nabla\delta\varphi\right] \\
& \left.-\frac{\mu^2+d_e^2}{\rho r^2}\left[(\mathcal{G}')^2|\nabla\delta\xi|^2 + \delta\xi\nabla(\mathcal{G}')^2\cdot\nabla\delta\xi\right]\right\} + \mathcal{Q}\,. \qquad (4.41)
\end{aligned}
$$

with $\mathcal{Q}$ given by

$$
\mathcal{Q} = \int d^2x\,(\delta\varphi\;\delta\xi\;\delta\rho)\,\mathcal{A}\,(\delta\varphi\;\delta\xi\;\delta\rho)^T\,, \qquad (4.42)
$$

where

$$
\mathcal{A} = \begin{pmatrix} A_{\varphi\varphi} & 0 & A_{\varphi\rho} \\ 0 & A_{\xi\xi} & A_{\xi\rho} \\ A_{\varphi\rho} & A_{\xi\rho} & A_{\rho\rho} \end{pmatrix}. \qquad (4.43)
$$

As regards the non-diagonal term, $-2r^{-2}\mathcal{F}'\mathcal{G}'\delta\varphi\delta\xi$, we choose to incorporate it into the purely positive and the diagonal parts of the functional, upon completing squares to reduce complexity in the subsequent analysis. By doing so we can write

$$
\delta^2\tilde{\mathcal{H}}_C = \int_D d^2x \quad \left\{\frac{|\nabla\delta\psi|^2}{r^2} + \frac{d_e^2r^2}{\rho}\left[\nabla\cdot\left(r^{-2}\nabla\delta\psi\right)\right]^2\right.
$$

$$+\rho\left(\delta v_\phi+\rho^{-1}v_\phi\delta\rho\right)^2+r^{-2}(\delta\mathcal{F}-\delta\mathcal{G})^2\Bigg\}+\tilde{\mathcal{Q}} \qquad (4.44)$$

and therefore $\tilde{\mathcal{Q}}>0$ suffices for stability. We have

$$\begin{aligned}\tilde{\mathcal{Q}} \;=\; & \mathcal{Q}-\int_D d^2x\left\{\frac{\gamma^2+d_e^2}{r^2\rho}\left[(\mathcal{F}')^2|\nabla\delta\varphi|^2\right.\right.\\ & \left.+2\mathcal{F}'\left(\delta\varphi\nabla\mathcal{F}'\cdot\nabla\delta\varphi-\rho^{-1}\delta\rho\nabla\mathcal{F}\cdot\nabla\delta\varphi\right)\right]\\ & +\frac{\mu^2+d_e^2}{r^2\rho}\left[(\mathcal{G}')^2|\nabla\delta\xi|^2+2\mathcal{G}'\left(\delta\xi\nabla\mathcal{G}'\cdot\nabla\delta\xi-\rho^{-1}\delta\rho\nabla\mathcal{G}\cdot\nabla\delta\xi\right)\right]\\ & +\left[2r^{-2}(\mathcal{F}')^2+\frac{\gamma^2+d_e^2}{\rho r^2}|\nabla\mathcal{F}'|^2\right](\delta\varphi)^2-2\frac{\gamma^2+d_e^2}{\rho^2r^2}\mathcal{F}'\mathcal{F}''|\nabla\varphi|^2\delta\rho\delta\varphi\\ & \left.+\left[2r^{-2}(\mathcal{G}')^2+\frac{\mu^2+d_e^2}{\rho r^2}|\nabla\mathcal{G}'|^2\right](\delta\xi)^2-2\frac{\mu^2+d_e^2}{\rho^2r^2}\mathcal{G}'\mathcal{G}''|\nabla\xi|^2\delta\rho\delta\xi\right\}. \qquad (4.45)\end{aligned}$$

Following [10], let us define the vectors $\mathbf{k}_\varphi:=\nabla\delta\varphi/\delta\varphi$, $\mathbf{k}_\xi:=\nabla\delta\xi/\delta\xi$. In view of this definition we can write (4.45) in the form (4.42) but in terms of a stability matrix $\tilde{\mathcal{A}}$ whose elements are given by

$$\begin{aligned}\tilde{A}_{\varphi\varphi}= & -\left(r^{-1}B_\phi^*+\gamma\Omega\right)\mathcal{F}''-\rho\mathcal{M}''-2r^{-2}(\mathcal{F}')^2\\ & -\frac{\gamma^2+d_e^2}{\rho r^2}\left[|\nabla\mathcal{F}'|^2+(\mathcal{F}')^2|\mathbf{k}_\varphi|^2+\mathbf{k}_\varphi\cdot\nabla(\mathcal{F}')^2\right], \qquad (4.46)\\ \tilde{A}_{\xi\xi}= & -\left(r^{-1}B_\phi^*+\mu\Omega\right)\mathcal{G}''-\rho\mathcal{N}''-2r^{-2}(\mathcal{G}')^2\\ & -\frac{\mu^2+d_e^2}{\rho r^2}\left[|\nabla\mathcal{G}'|^2+(\mathcal{G}')^2|\mathbf{k}_\xi|^2+\mathbf{k}_\xi\cdot\nabla(\mathcal{G}')^2\right], \qquad (4.47)\\ \tilde{A}_{\varphi\rho}= & -\mathcal{M}'+\frac{\gamma^2+d_e^2}{r^2\rho^2}\mathcal{F}'\left(\mathcal{F}''|\nabla\varphi|^2+\mathbf{k}_\varphi\cdot\nabla\mathcal{F}\right), \qquad (4.48)\\ \tilde{A}_{\xi\rho}= & -\mathcal{N}'+\frac{\mu^2+d_e^2}{r^2\rho^2}\mathcal{G}'\left(\mathcal{G}''|\nabla\xi|^2+\mathbf{k}_\xi\cdot\nabla\mathcal{G}\right), \qquad (4.49)\\ \tilde{A}_{\rho\rho}= & \rho^{-1}\Bigg[c_s^2-v_\phi^2-|\mathbf{v}_\perp|^2\\ & -\frac{d_e^2}{\rho^2}\left(r^2\left[\nabla\cdot\left(r^{-2}\nabla\psi\right)\right]^2+r^{-2}\left|\nabla(rB_\phi)\right|^2\right)\Bigg]. \qquad (4.50)\end{aligned}$$

Invoking the Cauchy-Schwartz inequality $\mathbf{k}_\varphi\cdot\nabla(\mathcal{F}')^2\le|\mathbf{k}_\varphi||\nabla(\mathcal{F}')^2|$ (and similarly for $\mathbf{k}_\xi$) it is clear that the following conditions

$$\begin{aligned}& -\left(r^{-1}B_\phi^*+\gamma\Omega\right)\mathcal{F}''-\rho\mathcal{M}''-2r^{-2}(\mathcal{F}')^2\\ & -\frac{\gamma^2+d_e^2}{\rho r^2}\left[|\nabla\mathcal{F}'|^2+(\mathcal{F}')^2|\mathbf{k}_\varphi|^2+|\mathbf{k}_\varphi||\nabla(\mathcal{F}')^2|\right]\\ & \equiv a_\varphi|\mathbf{k}_\varphi|^2+b_\varphi|\mathbf{k}_\varphi|+c_\varphi>0\,, \qquad (4.51)\\ & -\left(r^{-1}B_\phi^*+\mu\Omega\right)\mathcal{G}''-\rho\mathcal{N}''-2r^{-2}(\mathcal{G}')^2\end{aligned}$$

$$-\frac{\mu^2+d_e^2}{\rho r^2}\left[|\nabla\mathcal{G}'|^2+(\mathcal{G}')^2|\mathbf{k}_\xi|^2+|\mathbf{k}_\xi||\nabla(\mathcal{G}')^2|\right]$$
$$\equiv a_\xi|\mathbf{k}_\xi|^2+b_\xi|\mathbf{k}_\xi|+c_\xi>0\,, \tag{4.52}$$

are sufficient for $\tilde{A}_{\varphi\varphi}>0$ and $\tilde{A}_{\xi\xi}>0$ which are necessary for $\tilde{\mathcal{Q}}>0$. The two polynomials in $|\mathbf{k}_\varphi|$ and $|\mathbf{k}_\xi|$ must have at least one real positive root each. Given that $a_\varphi<0$, $b_\varphi<0$ and $a_\xi<0$, $b_\xi<0$, we understand that one root will be always negative; thus, in order for the second one to be positive, the products of the roots given by c_φ/a_φ, c_ξ/a_ξ, must be negative. Therefore we conclude that the conditions under which there exist exactly one real positive root for each polynomial are

$$c_\varphi := \quad -\left(r^{-1}B_\phi^*+\gamma\Omega\right)\mathcal{F}''-\rho\mathcal{M}''$$
$$-2r^{-2}(\mathcal{F}')^2-\frac{\gamma^2+d_e^2}{\rho r^2}|\nabla\mathcal{F}'|^2>0\,, \tag{4.53}$$
$$c_\xi := \quad -\left(r^{-1}B_\phi^*+\mu\Omega\right)\mathcal{G}''-\rho\mathcal{N}''$$
$$-2r^{-2}(\mathcal{G}')^2-\frac{\mu^2+d_e^2}{\rho r^2}|\nabla\mathcal{G}'|^2>0\,. \tag{4.54}$$

Now in view of (4.53)–(4.54) the two polynomials are also positive for $0\le|\mathbf{k}_\varphi|<k_\varphi^+$, $0\le|\mathbf{k}_\xi|<k_\xi^+$, where k_φ^+ and k_ξ^+ are the real roots of the polynomials in (4.51) and (4.52), respectively. This is true since the polynomials do not change sign within this domain and furthermore they are positive for $|\mathbf{k}_\varphi|=0$, $|\mathbf{k}_\xi|=0$. We thereby conclude that conditions (4.53) and (4.54) are sufficient for $\tilde{A}_{\varphi\varphi}>0$ and $\tilde{A}_{\xi\xi}>0$, if $|\mathbf{k}_\varphi|<k_\varphi^+$ and $|\mathbf{k}_\xi|<k_\xi^+$. On the other hand there is a topological lower bound on the admissible values of k_φ, k_ξ due to the Poincaré inequality,

$$\int_D d^2x\,|\mathbf{k}_\varphi|^2(\delta\varphi)^2=\int_D d^2x\,|\nabla\delta\varphi|^2\ge C^{-1}\int_D d^2x\,(\delta\varphi)^2\,, \tag{4.55}$$

where C is the Poincaré constant depending on the geometry of the domain D. Note that for smooth and bounded domains, the smallest eigenvalue of the Laplacian is an optimal value for C^{-1} since it minimizes the Rayleigh quotient. Lastly, inequality $\tilde{A}_{\varphi\varphi}\tilde{A}_{\xi\xi}\tilde{A}_{\rho\rho}-\tilde{A}_{\varphi\varphi}\tilde{A}_{\xi\rho}^2-\tilde{A}_{\xi\xi}\tilde{A}_{\varphi\rho}^2>0$ introduces additional restrictions e.g. condition $|\mathbf{v}|^2+d_e^2|\mathbf{J}|^2/\rho^2<c_s^2$ emerges as a necessary but not sufficient condition. Although we may possibly use similar manipulations with those employed above to arrive at sufficient conditions. Such a treatment will introduce additional constraints on the admissible equilibria and the values of $|\mathbf{k}_x|$, restricting the range of applicability of the resulting stability criterion, which will diverge even more from necessity. For this reason this analysis will not be pursued. Considering incompressible perturbations ($\delta\rho=0$), which are considered to be the most dangerous, the stability matrix is a 2×2 diagonal matrix with diagonal elements given by $\tilde{\mathcal{A}}_{\varphi\varphi}$ and $\tilde{\mathcal{A}}_{\xi\xi}$, thus leading to

the following sufficient conditional stability criterion

$$c_\varphi > 0\,, \quad c_\xi > 0\,,$$
$$\text{for} \quad |\mathbf{k}_\varphi| < k_\varphi^+\,,\; |\mathbf{k}_\xi| < k_\xi^+\,, \quad \langle \left(|\mathbf{k}_x|^2 - C^{-1}\right)(\delta x)^2 \rangle \geq 0\,, \tag{4.56}$$

where

$$k_x^+ = \frac{1}{2a_x}(-b_x - \sqrt{b_x^2 - 4a_x c_x})\,, \quad x = \varphi, \xi\,. \tag{4.57}$$

Note that the last inequality in (4.56) is satisfied for sure if $min(|\mathbf{k}_x|^2) \geq C^{-1}$ and hence, $c_x > 0\,,\ x = \varphi, \xi$, are sufficient stability conditions if $C^{-1} \leq |\mathbf{k}_x|^2 < k_x^+$. As a final point we stress that this stability criterion is general enough to capture a large variety of modes as long as k^+'s are large enough. Hence, it is practically useful to assess the stability of equilibria, when the equilibrium states under consideration render k^+'s as large as possible.

4.3 Dynamically accessible variations

Dynamical accessibility is concerned with variations that lie on symplectic leaves. Therefore, DAVs conserve the Casimirs, that is, $\delta\mathcal{C}_{da} = 0$, regardless the equilibrium conditions. Also, as seen in Chapter 1 the first order DAVs nullify the Hamiltonian on generic equilibrium points, including the energy-Casimir ones; thus

$$\delta\mathcal{H}[\boldsymbol{u}_e; \delta\boldsymbol{u}_{da}] = 0\,, \tag{4.58}$$

is a variational principle for generic equilibria. The sufficient stability criterion is provided by the positive definiteness of perturbation energy

$$\delta^2\mathcal{H}_{da}[\boldsymbol{u}_e] = \int d^3x \left(\frac{\delta^2\mathcal{H}}{\delta u^i \delta u^j}\bigg|_{\boldsymbol{u}_e} \delta u^i_{da} \delta u^j_{da} + \frac{\delta\mathcal{H}}{\delta u^i}\bigg|_{\boldsymbol{u}_e} \delta^2 u^i_{da} \right) \tag{4.59}$$

where δu_{da} and $\delta^2 u_{da}$ are, respectively, first and second order projections of arbitrary variations onto the symplectic leaves. The most efficient and rigorous methodology of producing these perturbations is upon using the Poisson bracket (see [12, 111, 8, 13]). Note that in the works of Arnold [11] and Isichenko [115] who used similar variations to study hydrodynamic and MHD stability, respectively, (called by them "isovortical" variations) they do not make use of the Poisson bracket approach. As described in the first chapter, this construction requires the introduction of a generating functional given by $\mathcal{W} = \int d^3x\, u_i \mathrm{g}^i$, where $\mathbf{g}$ is a state vector embodying the arbitrariness of the perturbations of the various dynamical variables. In view of these objects the DAVs to first order are given by $\delta \boldsymbol{u}_{da} = \{\boldsymbol{u}, \mathcal{W}\}$. In our case one has

$$\mathcal{W} = \int_V d^3x \, (\mathrm{g}_0 \rho + \mathbf{g}_1 \cdot \mathbf{v} + \mathbf{g}_2 \cdot \mathbf{B}^*) \, , \tag{4.60}$$

generating the following variations

$$\begin{aligned} \delta\rho_{da} &= \{\rho, \mathcal{W}\} = -\nabla \cdot \mathbf{g}_1 \, , & (4.61) \\ \delta\mathbf{v}_{da} &= \{\mathbf{v}, \mathcal{W}\} = -\nabla \mathrm{g}_0 + \rho^{-1}\mathbf{g}_1 \times \boldsymbol{\omega} + \rho^{-1}(\nabla \times \mathbf{g}_2) \times \mathbf{B}^* \, , & (4.62) \\ \delta\mathbf{B}^*_{da} &= \{\mathbf{B}^*, \mathcal{W}\} = \nabla \times \left[\rho^{-1}(\mathbf{g}_1 - d_i \nabla \times \mathbf{g}_2) \times \mathbf{B}^* \right. \\ &\qquad \left. + d_e^2 \rho^{-1}(\nabla \times \mathbf{g}_2) \times \boldsymbol{\omega}\right] . & (4.63) \end{aligned}$$

To show that the dynamically accessible variation of the Hamiltonian vanishes at general equilibria, we consider

$$\delta\mathcal{H}_{da} = \int_V d^3x \left[\rho \mathbf{v} \cdot \delta\mathbf{v}_{da} + \left(h + \frac{|\mathbf{v}|^2}{2} + d_e^2 \frac{|\mathbf{J}|^2}{2\rho}\right) \delta\rho_{da} + \mathbf{B} \cdot \delta\mathbf{B}^*_{da}\right] . \tag{4.64}$$

Substituting the expressions (4.61)–(4.63), performing integrations by part and omitting the surface integrals, we find

$$\begin{aligned} \delta\mathcal{H}_{da} = -\int_V d^3x \Bigg\{ & -\mathrm{g}_0 \nabla \cdot (\rho \mathbf{v}) \\ & + \mathbf{g}_1 \cdot \left[\mathbf{v} \times \boldsymbol{\omega} - \nabla \left(h + \frac{|\mathbf{v}|^2}{2} + d_e^2 \frac{|\mathbf{J}|^2}{2\rho}\right) + \frac{\mathbf{J} \times \mathbf{B}^*}{\rho}\right] \\ & + \mathbf{g}_2 \cdot \nabla \times \left[\mathbf{v} \times \mathbf{B}^* - d_i \frac{\mathbf{J} \times \mathbf{B}^*}{\rho} + d_e^2 \frac{\mathbf{J} \times \boldsymbol{\omega}}{\rho}\right] \Bigg\} . \end{aligned} \tag{4.65}$$

It is apparent that the coefficients of $g_0, \mathbf{g}_1, \mathbf{g}_2$ vanish in view of generic XMHD equilibrium conditions and consequently $\delta\mathcal{H}_{da}[\mathbf{u_e}] = 0$.

To proceed with stability analysis we need to calculate the second order variation of the Hamiltonian, which in view of Eq. (4.59) is

$$\begin{aligned} \delta^2\mathcal{H}_{da} = \int_V d^3x \Bigg\{ & \rho |\delta\mathbf{v}_{da}|^2 + \left(h + \frac{|\mathbf{v}|^2}{2} + d_e^2 \frac{|\mathbf{J}|^2}{2\rho^2}\right) \delta^2\rho_{da} \\ & + \left[h'(\rho) - d_e^2 \frac{|\mathbf{J}|^2}{\rho^3}\right] (\delta\rho_{da})^2 + 2\mathbf{v} \cdot \delta\mathbf{v}_{da} \delta\rho_{da} + \rho \mathbf{v} \cdot \delta^2 \mathbf{v}_{da} \\ & + \delta\mathbf{B}_{da} \cdot \delta\mathbf{B}^*_{da} + \mathbf{B} \cdot \delta^2\mathbf{B}^*_{da} + \frac{d_e^2}{\rho^2} \mathbf{J} \cdot \delta\mathbf{J}_{da} \delta\rho_{da} \Bigg\} , \end{aligned} \tag{4.66}$$

where $\delta\mathbf{J}_{da} = \nabla \times \mathbf{B}_{da}$. From the definition of $\mathbf{B}^*$ one has

$$\delta\mathbf{B}^*_{da} = \quad \delta\mathbf{B}_{da} - d_e^2 \nabla \times \left(\frac{\mathbf{J}}{\rho^2} \delta\rho_{da}\right) + d_e^2 \nabla \times \left(\frac{\delta\mathbf{J}_{da}}{\rho}\right) . \tag{4.67}$$

Upon inserting Eq. (4.67) into (4.66), the second term of (4.67) cancels out the last term in (4.66), leading to

$$\delta^2\mathcal{H}_{da} = \int_V d^3x \Big\{ \rho\big|\delta\mathbf{v}_{da} + \rho^{-1}\mathbf{v}\delta\rho_{da}\big|^2 + |\delta\mathbf{B}_{da}|^2$$
$$+d_e^2\frac{|\delta\mathbf{J}_{da}|^2}{\rho} + \rho^{-1}\left(c_s^2 - |\mathbf{v}|^2 - d_e^2\frac{|\mathbf{J}|^2}{\rho^2}\right)(\delta\rho_{da})^2 + \rho\mathbf{v}\cdot\delta^2\mathbf{v}_{da}$$
$$+\mathbf{B}\cdot\delta^2\mathbf{B}^*_{da} + \left(h + \frac{|\mathbf{v}|^2}{2} + d_e^2\frac{|\mathbf{J}|^2}{2\rho^2}\right)\delta^2\rho_{da}\Big\}. \tag{4.68}$$

Now, the second order variations of the field variables are given by

$$\delta^2\rho_{da} = 0, \tag{4.69}$$

$$\begin{aligned}
\delta^2\mathbf{v}_{da} &= \rho^{-1}\mathbf{g}_1\times\nabla\times\delta\mathbf{v}_{da} + \rho^{-1}(\nabla\times\mathbf{g}_2)\times\delta\mathbf{B}^*_{da} \\
&-\rho^{-2}[\mathbf{g}_1\times\boldsymbol{\omega} + (\nabla\times\mathbf{g}_2)\times\mathbf{B}^*]\delta\rho_{da} \\
&= \rho^{-1}(\boldsymbol{\zeta}\times\boldsymbol{\omega} + \boldsymbol{\eta}\times\mathbf{B}^*)\nabla\cdot(\rho\boldsymbol{\zeta}) + \boldsymbol{\zeta}\times\nabla\times(\boldsymbol{\zeta}\times\boldsymbol{\omega} + \boldsymbol{\eta}\times\mathbf{B}^*) \\
&+\boldsymbol{\eta}\times\nabla\times\left[(\boldsymbol{\zeta} - d_i\boldsymbol{\eta})\times\mathbf{B}^* + d_e^2\boldsymbol{\eta}\times\boldsymbol{\omega}\right],
\end{aligned} \tag{4.70}$$

$$\begin{aligned}
\delta^2\mathbf{B}^*_{da} &= \nabla\times\big\{\rho^{-1}(\mathbf{g}_1 - d_i\nabla\times\mathbf{g}_2)\times\delta\mathbf{B}^*_{da} + d_e^2\rho^{-1}(\nabla\times\mathbf{g}_2)\times\nabla\times\delta\mathbf{v}_{da} \\
&-\rho^{-2}[(\mathbf{g}_1 - d_i\nabla\times\mathbf{g}_2)\times\mathbf{B}^* + d_e^2(\nabla\times\mathbf{g}_2)\times\boldsymbol{\omega}]\delta\rho_{da}\big\} \\
&= \nabla\times\big\{(\boldsymbol{\zeta} - d_i\boldsymbol{\eta})\times\nabla\times\left[(\boldsymbol{\zeta} - d_i\boldsymbol{\eta})\times\mathbf{B}^* + d_e^2\boldsymbol{\eta}\times\boldsymbol{\omega}\right] \\
&+d_e^2\boldsymbol{\eta}\times\nabla\times(\boldsymbol{\zeta}\times\boldsymbol{\omega} + \boldsymbol{\eta}\times\mathbf{B}^*) \\
&+\rho^{-1}[(\boldsymbol{\zeta} - d_i\boldsymbol{\eta})\times\mathbf{B}^* + d_e^2\boldsymbol{\eta}\times\boldsymbol{\omega}]\nabla\cdot(\rho\boldsymbol{\zeta})\big\},
\end{aligned} \tag{4.71}$$

where $\boldsymbol{\zeta} := \rho^{-1}\mathbf{g}_1$ and $\boldsymbol{\eta} := \rho^{-1}\nabla\times\mathbf{g}_2$, introduced to facilitate comparisons with previous MHD and HMHD results [13, 15, 49]. Evidently equation $\nabla\cdot(\rho\boldsymbol{\eta}) = 0$ holds by definition of $\boldsymbol{\eta}$. After inserting expressions (4.69)–(4.71) into Eq. (4.66), and performing some straightforward manipulations we end up with

$$\delta^2\mathcal{H}_{da} = \int_V d^3x \Big\{ \rho\Big| -\nabla \mathrm{g}_0 + \boldsymbol{\zeta}\times\boldsymbol{\omega} + \boldsymbol{\eta}\times\mathbf{B}^* - \frac{\mathbf{v}}{\rho}\nabla\cdot(\rho\boldsymbol{\zeta})\Big|^2$$
$$+|\delta\mathbf{B}_{da}|^2 + d_e^2\rho^{-1}|\nabla\times\delta\mathbf{B}_{da}|^2 + \rho^{-1}\left(c_s^2 - |\mathbf{v}|^2 - d_e^2\frac{|\mathbf{J}|^2}{\rho^2}\right)[\nabla\cdot(\rho\boldsymbol{\zeta})]^2$$
$$-\boldsymbol{\zeta}\cdot\nabla\left(h + \frac{|\mathbf{v}|^2}{2} + d_e^2\frac{|\mathbf{J}|^2}{2\rho^2}\right)\nabla\cdot(\rho\boldsymbol{\zeta}) - \boldsymbol{\eta}\cdot(\mathbf{v}\times\mathbf{B}^*)\nabla\cdot(\rho\boldsymbol{\zeta})$$
$$-\rho^{-1}\boldsymbol{\eta}\cdot(-d_i\mathbf{J}\times\mathbf{B}^* + d_e^2\mathbf{J}\times\boldsymbol{\omega})\nabla\cdot(\rho\boldsymbol{\zeta}) - \rho(\boldsymbol{\zeta}\times\mathbf{v})\cdot\nabla\times(\boldsymbol{\zeta}\times\boldsymbol{\omega})$$
$$-\rho(\boldsymbol{\zeta}\times\mathbf{v})\cdot\nabla\times(\boldsymbol{\eta}\times\mathbf{B}^*) - \rho(\boldsymbol{\eta}\times\mathbf{v})\cdot\nabla\times[(\boldsymbol{\zeta} - d_i\boldsymbol{\eta})\times\mathbf{B}^*]$$
$$-[(\boldsymbol{\zeta} - d_i\boldsymbol{\eta})\times\mathbf{J}]\cdot\nabla\times[(\boldsymbol{\zeta} - d_i\boldsymbol{\eta})\times\mathbf{B}^*] - d_e^2\rho(\boldsymbol{\eta}\times\mathbf{v})\cdot\nabla\times(\boldsymbol{\eta}\times\boldsymbol{\omega})$$
$$-d_e^2[(\boldsymbol{\zeta} - d_i\boldsymbol{\eta})\times\mathbf{J}]\cdot\nabla\times(\boldsymbol{\eta}\times\boldsymbol{\omega}) - d_e^2(\boldsymbol{\eta}\times\mathbf{J})\cdot\nabla\times(\boldsymbol{\zeta}\times\boldsymbol{\omega})$$
$$-d_e^2(\boldsymbol{\eta}\times\mathbf{J})\cdot\nabla\times(\boldsymbol{\eta}\times\mathbf{B}^*)\Big\}. \tag{4.72}$$

The first term in the third line of (4.72), emerges from the substitution of $-\boldsymbol{\zeta}\cdot(\mathbf{J}\times\mathbf{B}^*)$ using the equilibrium momentum equation

$$-\mathbf{J}\times\mathbf{B}^* = \mathbf{v}\times\boldsymbol{\omega} - \nabla\left(h - \frac{|\mathbf{v}|^2}{2} - d_e^2\frac{|\mathbf{J}|^2}{2\rho^2}\right). \tag{4.73}$$

We can obtain the Hall and the Inertial MHD limits of (4.72) by setting $d_e = 0$ and $d_i = 0$, respectively. The Dirichlet stability theorem, the condition $\delta^2\mathcal{H}_{da} > 0$ $\forall\ \boldsymbol{\zeta}$, $\boldsymbol{\eta}$, g_0, with $\delta^2\mathcal{H}_{da}$ given by (4.72), ensures the stability of generic XMHD equilibria under dynamically accessible perturbations. As long as the variation of the magnetic field is treated as arbitrary, i.e., independent of $\boldsymbol{\zeta}$ and $\boldsymbol{\eta}$, even though it is not, the criterion is based on the positiveness of the terms that do not contain $\delta\mathbf{B}_{da}$. Thus, we understand that an improvement of this stability criterion can be obtained upon relating $\delta\mathbf{B}_{da}$ with $\boldsymbol{\zeta}$ and $\boldsymbol{\eta}$ by solving the differential equation that connects $\delta\mathbf{B}_{da}$ with $\delta\mathbf{B}^*_{da}(\boldsymbol{\zeta},\boldsymbol{\eta})$ and $\delta\rho_{da}(\boldsymbol{\zeta})$ and follows from definition (1.30). The solution can be effected by introducing a tensorial Green's function as follows

$$\begin{aligned}\delta\mathbf{B}_{da} = &\int_{V'} d^3x'\,\mathbf{G}(\mathbf{x}',\mathbf{x})\cdot\nabla\times\\ &\times\left[(\boldsymbol{\zeta}-d_i\boldsymbol{\eta})\times\mathbf{B}^* + d_e^2\boldsymbol{\eta}\times\boldsymbol{\omega} - d_e^2\frac{\mathbf{J}}{\rho^2}\nabla\cdot(\rho\boldsymbol{\zeta})\right],\end{aligned} \tag{4.74}$$

with $\mathbf{G}(\mathbf{x}',\mathbf{x})$ being the solution of

$$\left[1 + d_e^2\nabla\times\left(\frac{\nabla\times}{\rho}\right)\right]\mathbf{G}_i(\mathbf{x}',\mathbf{x}) = \boldsymbol{e}_i\delta(\mathbf{x}'-\mathbf{x}),\quad i=1,2,3. \tag{4.75}$$

For $\rho = const.$ things are simpler since the operator in the lhs of (4.75) becomes the Helmholtz operator (because $\nabla\cdot\delta\mathbf{B}_{da}=0$) and if Cartesian coordinates are employed then the equation splits into a set of three independent differential equations, one for each spatial component, so the Green's tensor can be replaced by a scalar Green's function which can be written as an infinite sum of Helmholtz basis functions. The problem remains though highly dependent on the particular boundary conditions.

As a simple application of the stability condition described above, let us consider a stationary axisymmetric equilibrium with purely toroidal flow, $\mathbf{v} = rv_\phi\nabla\phi$, and variations with perturbation vectors that never leave the surfaces $\psi^* = const.$, i.e. $\boldsymbol{\zeta}\cdot\nabla\psi^*=0$ and $\boldsymbol{\eta}\cdot\nabla\psi^*=0$, In this case the Lyapunov functional is reduced to (see Appendix C.1),

$$\begin{aligned}\delta^2\mathcal{H}_{da} = &\int_V d^3x\left\{\rho\left|\delta\mathbf{v}_{da} + \frac{rv_\phi}{\rho}\delta\rho_{da}\nabla\phi\right|^2 + |\delta\mathbf{B}_{da}|^2\right.\\ &\left.+d_e^2\rho^{-1}|\nabla\times\delta\mathbf{B}_{da}|^2 + \rho^{-1}\left(c_s^2 - v_\phi^2 - d_e^2\frac{|\mathbf{J}|^2}{\rho^2}\right)[\nabla\cdot(\rho\boldsymbol{\zeta})]^2\right\},\end{aligned} \tag{4.76}$$

and as a result, $c_s^2 - v_\phi^2 - d_e^2 \frac{|\mathbf{J}|^2}{\rho^2} > 0$ is sufficient for stability and also for the ellipticity of the equilibrium Grad-Shafranov-Bernoulli equations. Actually for the ellipticity of the equilibrium system, condition $c_s^2 - d_e^2 \frac{|\mathbf{J}_p|^2}{\rho^2} > 0$ is sufficient, as was shown in the previous chapter (see Eqs. (3.98) and (3.99)).

Having $\delta^2\mathcal{H}_{da}$ in form (4.72), it is difficult to compare with the corresponding HMHD and MHD expressions derived in [15] and [13] respectively. For this reason we reformulate the functional in (4.72) through some tedious but straightforward manipulations to obtain

$$\begin{aligned}
\delta^2\mathcal{H}_{da} = \int_V d^3x \Big\{ & \rho\big| -\nabla \mathrm{g}_0 + \boldsymbol{\zeta}\times\boldsymbol{\omega} + \boldsymbol{\eta}\times\mathbf{B}^* + \boldsymbol{\zeta}\cdot\nabla\mathbf{v} - \mathbf{v}\cdot\nabla\boldsymbol{\zeta}\big|^2 \\
& + \big|\delta\mathbf{B}_{da}\big|^2 - \rho\boldsymbol{\zeta}\cdot\nabla[h'(\rho)\nabla\cdot(\rho\boldsymbol{\zeta})] - (\boldsymbol{\zeta}\cdot\nabla h)\nabla\cdot(\rho\boldsymbol{\zeta}) - \boldsymbol{\zeta}\cdot(\mathbf{v}\cdot\nabla\mathbf{v})\nabla\cdot(\rho\boldsymbol{\zeta}) \\
& - (\boldsymbol{\zeta}\times\mathbf{J})\cdot\nabla\times(\boldsymbol{\zeta}\times\mathbf{B}^*) - \rho\boldsymbol{\zeta}\cdot[(\boldsymbol{\zeta}\cdot\nabla\mathbf{v} - \mathbf{v}\cdot\nabla\boldsymbol{\zeta})\cdot\nabla\mathbf{v}] \\
& - \rho\boldsymbol{\zeta}\cdot(\mathbf{v}\cdot\nabla)(\boldsymbol{\zeta}\cdot\nabla\mathbf{v} - \mathbf{v}\cdot\nabla\boldsymbol{\zeta}) + 2d_i(\boldsymbol{\zeta}\times\mathbf{J})\cdot\nabla\times(\boldsymbol{\eta}\times\mathbf{B}^*) \\
& - d_i\rho(\boldsymbol{\eta}\times\mathbf{B}^*)\cdot[\boldsymbol{\eta}\cdot\nabla(\mathbf{v} - d_i\mathbf{J}/\rho) - (\mathbf{v} - d_i\mathbf{J}/\rho)\cdot\nabla\boldsymbol{\eta}] \\
& + d_e^2\rho^{-1}\big|\nabla\times\delta\mathbf{B}_{da}\big|^2 - d_e^2\boldsymbol{\zeta}\cdot\nabla\left(\frac{|\mathbf{J}|^2}{2\rho^2}\right)\nabla\cdot(\rho\boldsymbol{\zeta}) + d_e^2\rho\boldsymbol{\zeta}\cdot\nabla\left[\frac{|\mathbf{J}|^2}{\rho^3}\nabla\cdot(\rho\boldsymbol{\zeta})\right] \\
& - d_e^2(\boldsymbol{\eta}\times\mathbf{J})\cdot\nabla\times(\boldsymbol{\eta}\times\mathbf{B}^*) - d_e^2\left[(2\boldsymbol{\zeta} - d_i\boldsymbol{\eta})\times\mathbf{J}\right]\cdot\nabla\times(\boldsymbol{\eta}\times\boldsymbol{\omega}) \\
& - d_e^2\rho(\boldsymbol{\eta}\times\mathbf{v})\cdot\nabla\times(\boldsymbol{\eta}\times\boldsymbol{\omega})\Big\}.
\end{aligned} \tag{4.77}$$

For obtaining (4.77) from (4.72) the vector identity $\nabla\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}\nabla\cdot\mathbf{b} - \mathbf{b}\nabla\cdot\mathbf{a} + \mathbf{b}\cdot\nabla\mathbf{a} - \mathbf{a}\cdot\nabla\mathbf{b}$ has been used. In addition the following relation was exploited

$$\begin{aligned}
\int d^3x\, \rho(\boldsymbol{\zeta}\times\mathbf{b})\cdot(\mathbf{a}\cdot\nabla\boldsymbol{\eta} - \boldsymbol{\eta}\cdot\nabla\mathbf{a}) &= \int d^3x\, \rho(\boldsymbol{\eta}\times\mathbf{b})\cdot(\mathbf{a}\cdot\nabla\boldsymbol{\zeta} - \boldsymbol{\zeta}\cdot\nabla\mathbf{a}) \\
&+ \int d^3x\, \{(\boldsymbol{\eta}\times\boldsymbol{\zeta})\cdot[\rho\nabla\times(\mathbf{a}\times\mathbf{b}) - \rho\mathbf{a}\nabla\cdot\mathbf{b} - \mathbf{b}\nabla\cdot(\rho\mathbf{a})]\}\,.
\end{aligned} \tag{4.78}$$

A proof for this relation is provided in Appendix C.2.

It becomes clear that the case $d_e = 0$ corresponds to the barotropic counterpart of the HMHD $\delta^2\mathcal{H}_{da}$ given in [15], while if we further impose $d_i = 0$ we find $\delta^2\mathcal{H}_{da} = \int_V d^3x\, \rho|\delta\mathbf{v}_{da} + \boldsymbol{\zeta}\cdot\nabla\mathbf{v} - \mathbf{v}\cdot\nabla\boldsymbol{\zeta}|^2 + \delta W$, where δW is the Frieman-Rotenberg expression for potential energy [23], consistent with the results found in [13, 49]. The correct MHD limit of (4.77) reveals an important advantage of the dynamically accessible method compared to the energy-Casimir one. As it has been highlighted in Chapter 2 and also in [71, 73, 89], the MHD limit of the Casimirs and variational functionals (e.g. the Lagrangian) of XMHD and HMHD, presents certain peculiarities because the Hall term gives rise to singular perturbations, making the derivation of their MHD counterparts rather not straightforward. As regards the Casimirs, a detailed analysis leading to their correct MHD limit is presented in Chapter 2. One can also consult the references [71] and [89]. Hence, it is natural that this complication is inherited by the variational principles involving the Casimirs, e.g. the energy-Casimir method. However, in deriving $\delta^2\mathcal{H}_{da}$ we did not make use of the Casimirs, and therefore their problematic MHD limit does not affect the corresponding limit of the dynamically accessible stability criterion.

4.4 Perturbations in mixed Eulerian-Lagrangian framework

As mentioned in Sec. 1.4, within tthe Lagrangian framework the fluids are not described in terms of vector fields measured at fixed position $\mathbf{x} \in \mathbb{R}^3$ as in the Eulerian framework adopted above, but in terms of Lagrangian (or material) variables suitable for tracking the motion of the individual fluid elements. The material variables are the positions of fluid elements at given instant: $\mathbf{q}_s(\mathbf{a}_s, t)$ ($s = i, e$ standing for the ion and electron species) where $\mathbf{a}_s \in \mathbb{R}^3$ are the fluid element labels, usually taken as the element's position at $t = 0$. For a two-fluid theory, which is the starting point of the XMHD model, the Lagrange-Euler map (1.67), written for each one of the constituent fluids is

$$\mathbf{v}_s(\mathbf{x}, t) = \dot{\mathbf{q}}_s(\mathbf{a}_s, t)\Big|_{\mathbf{a}_s=\mathbf{q}_s^{-1}(\mathbf{x},t)}, \tag{4.79}$$

$$n_s(\mathbf{x}, t) = \frac{n_{s0}(\mathbf{a}_s)}{\mathcal{J}_s(\mathbf{a}_s, t)}\Big|_{\mathbf{a}_s=\mathbf{q}_s^{-1}(\mathbf{x},t)}, \tag{4.80}$$

$$s_s(\mathbf{x}, t) = s_{s0}(\mathbf{a}_s)\big|_{\mathbf{a}_s=\mathbf{q}_s^{-1}(\mathbf{x},t)}, \tag{4.81}$$

where s_s are the specific entropies of the fluids and $\mathcal{J}_s$ ($s = i, e$), are the Jacobians of $\mathbf{q}_s$ with respect to $\mathbf{a}_s$, i.e. $\mathcal{J}_s := det(\partial q_s^i/\partial a_s^j)$. For barotropic fluids, s_s are just constants. The difference between the single-fluid MHD and the two-fluid case is that in the former model the magnetic field can be expressed in terms of Lagrangian variables, due to the frozen-in property of the magnetic field lines. In the case of HMHD and XMHD one can find similar frozen-in properties [20] as well. However, in XMHD this property concerns generalized magnetic-vorticity fields and as a result only the field $\mathbf{B}^*$ can be explicitly expressed in terms of the Lagrangian variables. This means that similar expressions for $\mathbf{B}$ can be found only implicitly through a relation similar to (4.74). This makes a fully Lagrangian description of the XMHD model more involved and less universal than the corresponding description for the MHD, since it requires the solution of a differential equation for $\mathbf{B}$, which depends on the specific boundary conditions. Another peculiarity is that in a fully Lagrangian description the usual Legendre transform cannot be performed and therefore one need to start with a phase-space Lagrangian [20]. One way to get rid of those peculiarities is to sacrifice some information about the relationship of the magnetic field with the fluid motion, describing the former as an independent Eulerian variable. Despite this compromise, the resulting mixed Eulerian-Lagrangian description [116], is still sufficient in order to perform stability analyses and make comparisons with other stability methods.

To perform a stability analysis in terms of Lagrangian displacements, within a fully Lagrangian framework, as in the work of Newcomb [117] for MHD or a mixed Eulerian-Lagrangian framework as was done by Vuilemin [118] for the complete two-fluid model (without quasineutrality), we need to start with the Lagrangian of the model and compute its second order variation induced by small perturbations. The two-fluid Lagrangian with Maxwell's term being neglected in view of the assumption $v_A \ll c$ (v_A and c are the Alfvén speed and the speed of light, respectively) [116] is

$$\mathcal{L} = \sum_s \int d^3 \mathrm{a}_s \Big\{ \frac{1}{2} m_s n_{s0}(\mathbf{a}_s) \big|\dot{\mathbf{q}}_s(\mathbf{a}_s,t)\big|^2 - m_s n_{s0}(\mathbf{a}_s) U_s \left(s_s, \frac{m_s n_{s0}(\mathbf{a}_s)}{\mathcal{J}_s(\mathbf{a}_s,t)} \right) \\ + \int d^3x\, \delta(\mathbf{x} - \mathbf{q}_s(\mathbf{a}_s,t)) e_s n_{s0}(\mathbf{a}_s) \left[\dot{\mathbf{q}}_s(\mathbf{a}_s,t) \cdot \mathbf{A}(\mathbf{x},t) - \Phi(\mathbf{x},t) \right] \Big\} \\ - \frac{1}{2\mu_0} \int d^3x \big|\nabla \times \mathbf{A}(\mathbf{x},t)\big|^2 , \tag{4.82}$$

where $\mathbf{A}$ and Φ are the vector and electrostatic potentials, respectively. Let us impose Lagrangian particle density homogeneity, i.e. the assumption that any fluid element, belonging to either the ion or the electron fluid, contains the same number of particles, that is $n_{i0}(\mathbf{a}_i) = n_{e0}(\mathbf{a}_e) = n_0 = constant$. In view of (4.80), the assumption of Lagrangian homogeneity along with the imposition of Eulerian quasineutrality $n_i(\mathbf{x},t) = n_e(\mathbf{x},t)$ leads to

$$\mathcal{J}_i\big|_{\mathbf{a}_i = \mathbf{q}_i^{-1}(\mathbf{x},t)} = \mathcal{J}_e\big|_{\mathbf{a}_e = \mathbf{q}_e^{-1}(\mathbf{x},t)} . \tag{4.83}$$

Now, since the trajectories $\mathbf{q}_i$, $\mathbf{q}_e$ of the ion and electron fluid elements are in general different, then at time $t > 0$ they will be located at different positions $\mathbf{x}$, $\mathbf{x}'$ unless the fluid elements $\mathbf{a}_i$ and $\mathbf{a}_e$ are chosen appropriately to make $\mathbf{x}' = \mathbf{x}$. Therefore, in general, if we try to write down a single fluid version of (4.82) and then take the Lagrange-Euler map, we will end up with a nonlocal Lagrangian in the Eulerian description. Locality on the Eulerian level has to be imposed. This is equivalent to matching up the ion and electron fluid elements on the basis of the map $\mathbf{a}_e = \mathbf{q}_e^{-1}(\mathbf{q}_i(\mathbf{a}_i,t),t)$, as it is schematically shown in Fig. 4.3 (see also the corresponding explanation in [20]). The final step for obtaining an XMHD action is to substitute the ion and electron Lagrangian variables with XMHD-like variables, which would play the role of Lagrangian analogues for $\mathbf{v}$ and $\mathbf{J}/(en)$. In this regard we define two new Lagrangian quantities $\mathbf{Q}$ and $\mathbf{D}$ through the following relations:

$$\mathbf{Q}(\mathbf{a}_i,\mathbf{a}_e,t) := \frac{m_i}{m}\mathbf{q}_i(\mathbf{a}_i,t) + \frac{m_e}{m}\mathbf{q}_e(\mathbf{a}_e,t) , \tag{4.84}$$

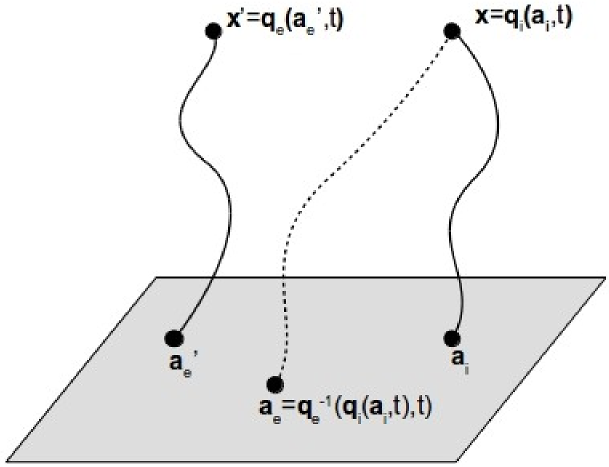


FIGURE 4.3: The flow trajectories of a random pair of electron and ion fluid elements labeled by $\mathbf{a}_e'$ and $\mathbf{a}_i$, respectively, end up at different locations at time $t > 0$. However, if the electron label is chosen so as $\mathbf{a}_e = \mathbf{q}_e^{-1}(\mathbf{q}_i(\mathbf{a}_i, t))$ then the trajectories intersect at time $t > 0$.

$$\mathbf{D}(\mathbf{a}_i, \mathbf{a}_e, t) := \mathbf{q}_i(\mathbf{a}_i, t) - \mathbf{q}_e(\mathbf{a}_e, t)\,. \tag{4.85}$$

The inverse transformation reads as follows:

$$\begin{aligned} \mathbf{q}_s(\mathbf{a}_s, t) &:= \mathbf{Q}(\mathbf{a}_s, \mathbf{a}_{s'}, t)\big|_{\mathbf{a}_{s'}=\mathbf{q}_{s'}^{-1}(\mathbf{q}_s(\mathbf{a}_s,t),t)} \\ &+\alpha_s \mathbf{D}(\mathbf{a}_s, \mathbf{a}_{s'}, t)\big|_{\mathbf{a}_{s'}=\mathbf{q}_{s'}^{-1}(\mathbf{q}_s(\mathbf{a}_s,t),t)}\,, \quad s' \neq s\,, \end{aligned} \tag{4.86}$$

where $\alpha_i = m_e/m$ and $\alpha_e = -m_i/m$. Now we are in position to write down the XMHD Lagrangian in $(\mathbf{Q}, \mathbf{D})$ variables and imposing locality

$$\begin{aligned} \mathcal{L} = & \int\!\!\int d^3\mathrm{a}_i d^3\mathrm{a}_e \delta(\mathbf{a}_e - \mathbf{q}_e^{-1}(\mathbf{q}_i(\mathbf{a}_i, t), t)) \times \\ & \times n_0 \sum_{s=i,e} \left[\frac{m_s}{2} \big|\dot{\mathbf{Q}}(\mathbf{a}_s, \mathbf{a}_{s'}, t)\big|^2 + \frac{m_s}{2}\alpha_s^2 \big|\dot{\mathbf{D}}(\mathbf{a}_s, \mathbf{a}_{s'}, t)\big|^2 \right. \\ & + e_s \dot{\mathbf{Q}}(\mathbf{a}_s, \mathbf{a}_{s'}, t) \cdot \mathbf{A}(\mathbf{q}_s(\mathbf{a}_s, t), t) - e_s \Phi(\mathbf{q}_s(\mathbf{a}_s, , t), t) \\ & \left. + e_s \alpha_s \dot{\mathbf{D}}(\mathbf{a}_s, \mathbf{a}_{s'}, t) \cdot \mathbf{A}(\mathbf{q}_s(\mathbf{a}_s, t), t) - m_s U_s \left(s_s, \frac{m_s n_0}{\mathcal{J}_s(\mathbf{a}_s, t)} \right) \right] \\ & - \frac{1}{2\mu_0} \int d^3x \big|\nabla \times \mathbf{A}(\mathbf{x}, t)\big|^2\,,\ s' \neq s. \end{aligned} \tag{4.87}$$

In general we are interested in examining the stability of stationary equilibria in the Eulerian picture. It is well known [8, 49, 117] that not all Eulerian equilibria correspond to Lagrangian ones e.g. for an Eulerian equilibrium state with flow an infinite number of fluid elements have to be in motion for the realization of this flow. However, in the Lagrangian framework, moving fluid elements correspond to time dependent material variables. Therefore, we conclude that stationary Eulerian states correspond to time dependent Lagrangian trajectories $\mathbf{q}_{s0} = \mathbf{q}_{s0}(\mathbf{a}_s, t)$. So, we expand the material variables around time dependent reference trajectories considering a small perturbation, Hence the fields should be decomposed as follows

$$\mathbf{Q}(\mathbf{a}_i, \mathbf{a}_e, t) = \mathbf{Q}_0(\mathbf{a}_i, \mathbf{a}_e, t) + \boldsymbol{\zeta}(\mathbf{a}_i, \mathbf{a}_e, t)\,, \tag{4.88}$$

$$\mathbf{D}(\mathbf{a}_i, \mathbf{a}_e, t) = \mathbf{D}_0(\mathbf{a}_i, \mathbf{a}_e, t) + \boldsymbol{\eta}(\mathbf{a}_i, \mathbf{a}_e, t)\,, \tag{4.89}$$

$$\mathbf{A}(\mathbf{x}, t) = \mathbf{A}_0(\mathbf{x}) + \mathbf{A}_1(\mathbf{x}, t)\,, \tag{4.90}$$

$$\Phi(\mathbf{x}, t) = \Phi_0(\mathbf{x}) + \Phi_1(\mathbf{x}, t)\,, \tag{4.91}$$

where the quantities with subscript 0 define an equilibrium state, those with subscript 1 represent the perturbed electromagnetic field and $\boldsymbol{\zeta}$, $\boldsymbol{\eta}$ are Lagrangian displacements accounting for the perturbation of the fluid element trajectories. The $\mathbf{q}$'s in the delta function in (4.87) need not be expanded because the ion and electron fluid elements can be paired at equilibrium; $\delta(\mathbf{a}_e - \mathbf{q}_e^{-1}(\mathbf{q}_i(\mathbf{a}, t), t))$ can be replaced by $\delta(\mathbf{a}_e - \mathbf{q}_{e0}^{-1}(\mathbf{q}_{i0}(\mathbf{a}, t), t))$. Since we have a continuum, every perturbed position $\mathbf{q}_s(\mathbf{a}_s, t)$ always correspond to the unperturbed position of another pair and all pairs are taken into account because the integrals run over the whole fluid domain. Hence, in view of (4.88)–(4.91) we find, using (4.87), a perturbed Lagrangian, i.e. $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 + \ldots$. For stability we are interested in $\mathcal{L}_2$ because it describes the linearized dynamics, while $\mathcal{L}_0$ is merely a constant and $\mathcal{L}_1$ vanishes at equilibrium. To write down the second order perturbation of the Lagrangian we need to expand the electromagnetic potentials and the internal energies. The magnetic and electric potentials are computed on the fluid trajectories, thus, up to second order, they are

$$\begin{aligned}\mathbf{A}(\mathbf{q}_{s0} + \boldsymbol{\zeta} + \alpha_s\boldsymbol{\eta}, t) &= \mathbf{A}_0(\mathbf{q}_{s0}) + \mathbf{A}_1(\mathbf{q}_{s0}, t) + (\boldsymbol{\zeta} + \alpha_s\boldsymbol{\eta})\cdot\nabla_{\mathbf{q}_s}\mathbf{A}_0(\mathbf{q}_{s0})\\ &+(\boldsymbol{\zeta} + \alpha_s\boldsymbol{\eta})\cdot\nabla_{\mathbf{q}_s}\mathbf{A}_1(\mathbf{q}_{s0}, t) + \frac{1}{2}(\boldsymbol{\zeta} + \alpha_s\boldsymbol{\eta})(\boldsymbol{\zeta} + \alpha_s\boldsymbol{\eta}) : \nabla_{\mathbf{q}_s}\nabla_{\mathbf{q}_s}\mathbf{A}_0(\mathbf{q}_{s0})\,,\end{aligned} \tag{4.92}$$

$$\begin{aligned}\Phi(\mathbf{q}_{s0} + \boldsymbol{\zeta} + \alpha_s\boldsymbol{\eta}, t) &= \Phi_0(\mathbf{q}_{s0}) + \Phi_1(\mathbf{q}_{s0}, t) + (\boldsymbol{\zeta} + \alpha_s\boldsymbol{\eta})\cdot\nabla_{\mathbf{q}_s}\Phi_0(\mathbf{q}_{s0})\\ &+(\boldsymbol{\zeta} + \alpha_s\boldsymbol{\eta})\cdot\nabla_{\mathbf{q}_s}\Phi_1(\mathbf{q}_{s0}, t) + \frac{1}{2}(\boldsymbol{\zeta} + \alpha_s\boldsymbol{\eta})(\boldsymbol{\zeta} + \alpha_s\boldsymbol{\eta}) : \nabla_{\mathbf{q}_s}\nabla_{\mathbf{q}_s}\Phi_0(\mathbf{q}_{s0})\,,\end{aligned} \tag{4.93}$$

where $\mathbf{ab} : \mathbf{cd} := a_i b_j c^j d^i$. The second order perturbative expansion of the internal energy terms is performed along lines similar to those of the single fluid case (see [8]). The difficulty in this expansion is that the Jacobians contain a dependence on the gradients of the fluid trajectories, therefore we need to know how to differentiate the $\mathcal{J}$'s, since the expansion of the internal energy is effected through

$$\mathcal{J}_s = \mathcal{J}_{s0} + \frac{\partial \mathcal{J}_s}{\partial q^i_{s,j}}\frac{\partial \zeta^i_s}{\partial a^j_s} + \frac{1}{2}\frac{\partial^2 \mathcal{J}_s}{\partial q^i_{s,k}\partial q^j_{s,\ell}}\frac{\partial \zeta^i_s}{\partial a^k_s}\frac{\partial \zeta^j_s}{\partial a^\ell_s}\,, \tag{4.94}$$

where $q^i_{s,j} := \frac{\partial q^i_s}{\partial a^j_s}$. The derivatives of the Jacobian are $\frac{\partial \mathcal{J}_s}{\partial q^i_{s,j}} = C_{si}^{\ j}$, where $C_{si}^{\ j} = \frac{1}{2}\epsilon_{i\ell k}\epsilon^{jmn}\frac{\partial q^\ell_s}{\partial a^m_s}\frac{\partial q^k_s}{\partial a^n_s}$ are the cofactors of $\partial q^i_s/\partial a^j_s$ in $\mathcal{J}_s$. Following the procedure in [8] and [117] we find

$$\mathcal{J}_{s1} = \mathcal{J}_{s0}\frac{\partial \zeta^i_s}{\partial q^i_s}\,, \quad \mathcal{J}_{s2} = \frac{\mathcal{J}_{s0}}{2}\left[\left(\frac{\partial \zeta^i_s}{\partial q^i_s}\right)^2 - \frac{\partial \zeta^i_s}{\partial q^j_s}\frac{\partial \zeta^j_s}{\partial q^i_s}\right]. \tag{4.95}$$

With the second order perturbative expansion of the Jacobians at hand we can find the second order perturbation of the internal energies in terms of the displacement vectors as follows

$$U_{s2} = \frac{n_0}{2\mathcal{J}_{s0}}\left\{\frac{\partial U_s}{\partial n}\left[\left(\frac{\partial \zeta^i}{\partial q^i_s} + \alpha_s\frac{\partial \eta^i}{\partial q^i_s}\right)^2 + \left(\frac{\partial \zeta^i}{\partial q^j_s} + \alpha_s\frac{\partial \eta^i}{\partial q^j_s}\right)\left(\frac{\partial \zeta^j}{\partial q^i_s} + \alpha_s\frac{\partial \eta^j}{\partial q^i_s}\right)\right]\right.$$
$$\left. + \frac{n_0}{\mathcal{J}_{s0}}\frac{\partial^2 U_s}{\partial n^2}\left(\frac{\partial \zeta^i}{\partial q^i_s} + \alpha_s\frac{\partial \eta^i}{\partial q^i_s}\right)^2\right\}. \tag{4.96}$$

This expression can be deduced upon expanding U_s as follows:

$$U_s = U_s\left(s_s, \frac{m_s n_0}{\mathcal{J}_{s0}}\right) - \frac{n_0}{(\mathcal{J}_{s0})^2}\frac{\partial U_s}{\partial n}\mathcal{J}_{s1} - \frac{n_0}{(\mathcal{J}_{s0})^2}\frac{\partial U_s}{\partial n}\mathcal{J}_{s2}$$
$$+ \frac{n_0}{(\mathcal{J}_{s0})^3}\frac{\partial U_s}{\partial n}(\mathcal{J}_{s1})^2 + \frac{1}{2}\frac{n_0^2}{(\mathcal{J}_{s0})^4}\frac{\partial^2 U_s}{\partial n^2}(\mathcal{J}_{s1})^2\,, \tag{4.97}$$

and then substituting (4.95). Henceforth, the subscript 0 will be dropped on the understanding that from now on $\mathbf{A}, \Phi$, $\mathbf{Q}$, $\mathbf{D}$, $\mathbf{q}_s$ and $\mathcal{J}_s$ correspond to equilibrium. Using the results (4.92)–(4.93) and (4.96) we are able to construct $\mathcal{L}_2$

$$\begin{aligned}\mathcal{L}_2 \quad &= \int\!\!\int d^3\mathrm{a}_i d^3\mathrm{a}_e \delta(\mathbf{a}_e - \mathbf{q}_e^{-1}(\mathbf{q}_i(\mathbf{a}_i,t),t))n_0\sum_s\left\{\frac{m_s}{2}|\dot{\boldsymbol{\zeta}}|^2 + \alpha_s^2\frac{m_s}{2}|\dot{\boldsymbol{\eta}}|^2\right.\\
&+e_s\left(\dot{\mathbf{Q}} + \alpha_s\dot{\mathbf{D}}\right)\cdot\big[\,(\boldsymbol{\zeta} + \alpha_s\boldsymbol{\eta})\cdot\nabla_{\mathbf{q}_s}\mathbf{A}_1(\mathbf{q}_s,t)\\
&+\frac{1}{2}(\boldsymbol{\zeta} + \alpha_s\boldsymbol{\eta})(\boldsymbol{\zeta} + \alpha_s\boldsymbol{\eta}):\nabla_{\mathbf{q}_s}\nabla_{\mathbf{q}_s}\mathbf{A}(\mathbf{q}_s)\big]\\
&+e_s(\dot{\boldsymbol{\zeta}} + \alpha_s\dot{\boldsymbol{\eta}})\cdot\big[\mathbf{A}_1(\mathbf{q}_s,t) + (\boldsymbol{\zeta} + \alpha_s\boldsymbol{\eta})\cdot\nabla_{\mathbf{q}_s}\mathbf{A}(\mathbf{q}_s)\big]\\
&-e_s(\boldsymbol{\zeta} + \alpha_s\boldsymbol{\eta})\cdot\nabla_{\mathbf{q}_s}\Phi_1(\mathbf{q}_s,t) - \frac{e_s}{2}(\boldsymbol{\zeta} + \alpha_s\boldsymbol{\eta})(\boldsymbol{\zeta} + \alpha_s\boldsymbol{\eta}):\nabla_{\mathbf{q}_s}\nabla_{\mathbf{q}_s}\Phi(\mathbf{q}_s)\\
&-\frac{n_0^2}{2\mathcal{J}_s^2}\frac{\partial^2\mathcal{U}_s}{\partial n^2}(\nabla_{\mathbf{q}_s}\cdot\boldsymbol{\zeta} + \alpha_s\nabla_{\mathbf{q}_s}\cdot\boldsymbol{\eta})^2 - \frac{n_0}{2\mathcal{J}_s}\frac{\partial\mathcal{U}_s}{\partial n}\big[\,(\nabla_{\mathbf{q}_s}\cdot\boldsymbol{\zeta} + \alpha_s\nabla_{\mathbf{q}_s}\cdot\boldsymbol{\eta})^2\\
&\left.+\nabla_{\mathbf{q}_s}(\boldsymbol{\zeta} + \alpha_s\boldsymbol{\eta}):\nabla_{\mathbf{q}_s}(\boldsymbol{\zeta} + \alpha_s\boldsymbol{\eta})\,\big]\right\} - \frac{1}{2\mu_0}\int d^3x\big|\nabla\times\mathbf{A}_1(\mathbf{x},t)\big|^2\,,\end{aligned} \tag{4.98}$$

where $\mathcal{U}_s = m_s U_s$ and $\nabla_{\mathbf{q}_s} \equiv \nabla_{\mathbf{q}_{s0}}$. Result (4.98) is not very different from the result of Vuilemin [118]; actually, it is the quasineutral counterpart of his second order perturbed Lagrangian, written however in terms of the XMHD Lagrangian displacements

$\boldsymbol{\zeta}$, $\boldsymbol{\eta}$ instead of the two-fluid ones $\boldsymbol{\xi}_i$, $\boldsymbol{\xi}_e$. Moreover, (4.98) is applicable for generic thermodynamic closures with scalar pressure, not only for fluids obeying the adiabatic ideal-gas law as in [118]. The most important advantage of our formulation can be seen though, after employing the Lagrange-Euler map: firstly because (4.98) explicitly dictates how the labels of the fluid elements are related so that the Lagrange-Euler map will result in a local Lagrangian and secondly because its Eulerian counterpart will be expressed in terms of the MHD-like variables, namely $\mathbf{v}$ and $\mathbf{J}$.

To employ the Lagrange-Euler map we need to "Eulerianize" the displacement vectors. Let us begin with the Lagrange-Euler map and its inverse in order to understand how $\mathbf{Q}$, $\mathbf{D}$ and the displacements $\boldsymbol{\zeta}$, $\boldsymbol{\eta}$ are mapped in the Eulerian coordinates. From (4.79) and (4.84)–(4.86) we can effectively construct every map we need. For example

$$\dot{\mathbf{Q}}(\mathbf{a}_i, \mathbf{a}_e, t) = \frac{m_i}{m}\left(\mathbf{v} + \frac{m_e}{men}\mathbf{J}\right)\bigg|_{\mathbf{x}=\mathbf{q}_i(\mathbf{a}_i,t)} + \frac{m_e}{m}\left(\mathbf{v} - \frac{m_i}{men}\mathbf{J}\right)\bigg|_{\mathbf{x}=\mathbf{q}_e(\mathbf{a}_e,t)},$$
$$\dot{\mathbf{D}}(\mathbf{a}_i, \mathbf{a}_e, t) = \left(\mathbf{v} + \frac{m_e}{men}\mathbf{J}\right)\bigg|_{\mathbf{x}=\mathbf{q}_i(\mathbf{a}_i,t)} - \left(\mathbf{v} - \frac{m_i}{men}\mathbf{J}\right)\bigg|_{\mathbf{x}=\mathbf{q}_e(\mathbf{a}_e,t)}. \quad (4.99)$$

If these expressions are computed at $\mathbf{a}_e = \mathbf{q}_e^{-1}(\mathbf{q}_i(\mathbf{a}_i, t), t)$ as in the Lagrangian (4.87) at equilibrium we have

$$\dot{\mathbf{Q}}_0(\mathbf{a}_i, t) = \mathbf{v}(\mathbf{x})\big|_{\mathbf{x}=\mathbf{q}_{i0}(\mathbf{a}_i,t)}, \quad \text{and} \quad \dot{\mathbf{D}}_0(\mathbf{a}_i, t) = e^{-1}n^{-1}(\mathbf{x})\mathbf{J}(\mathbf{x})\big|_{\mathbf{x}=\mathbf{q}_{i0}(\mathbf{a}_i,t)}.$$

For the Eulerianization of the displacement vectors we define their Eulerian displacements $\tilde{\boldsymbol{\zeta}}$ and $\tilde{\boldsymbol{\eta}}$ by

$$\begin{aligned}\boldsymbol{\zeta}(\mathbf{a}_i, \mathbf{a}_e, t) &= \frac{m_i}{m}\left[\tilde{\boldsymbol{\zeta}}(\mathbf{x}, t) + \frac{m_e}{m}\tilde{\boldsymbol{\eta}}(\mathbf{x}, t)\right]_{\mathbf{x}=\mathbf{q}_{i0}(\mathbf{a}_i,t)} \\ &\quad + \frac{m_e}{m}\left[\tilde{\boldsymbol{\zeta}}(\mathbf{x}, t) - \frac{m_i}{m}\tilde{\boldsymbol{\eta}}(\mathbf{x}, t)\right]_{\mathbf{x}=\mathbf{q}_{e0}(\mathbf{a}_e,t)}, \\ \boldsymbol{\eta}(\mathbf{a}_i, \mathbf{a}_e, t) &= \left[\tilde{\boldsymbol{\zeta}}(\mathbf{x}, t) + \frac{m_e}{m}\tilde{\boldsymbol{\eta}}(\mathbf{x}, t)\right]_{\mathbf{x}=\mathbf{q}_{i0}(\mathbf{a}_i,t)} \\ &\quad - \left[\tilde{\boldsymbol{\zeta}}(\mathbf{x}, t) - \frac{m_i}{m}\tilde{\boldsymbol{\eta}}(\mathbf{x}, t)\right]_{\mathbf{x}=\mathbf{q}_{e0}(\mathbf{a}_e,t)}. \end{aligned} \quad (4.100)$$

Taking the time derivatives of (4.100) with $\mathbf{a}_i$ and $\mathbf{a}_e$ held constant, we find

$$\begin{aligned}\dot{\boldsymbol{\zeta}}(\mathbf{a}_i, \mathbf{a}_e, t) &= \partial_t\tilde{\boldsymbol{\zeta}}(\mathbf{x}, t) + \mathbf{v}\cdot\nabla\tilde{\boldsymbol{\zeta}}(\mathbf{x}, t) + \frac{m_i m_e}{m^2}\boldsymbol{w}\cdot\nabla\tilde{\boldsymbol{\eta}}(\mathbf{x}, t), \\ \dot{\boldsymbol{\eta}}(\mathbf{a}_i, \mathbf{a}_e, t) &= \partial_t\tilde{\boldsymbol{\eta}}(\mathbf{x}, t) + \mathbf{v}\cdot\nabla\tilde{\boldsymbol{\eta}}(\mathbf{x}, t) \\ &\quad + \boldsymbol{w}\cdot\nabla\tilde{\boldsymbol{\zeta}}(\mathbf{x}, t) + \frac{m_e^2 - m_i^2}{m^2}\boldsymbol{w}\cdot\nabla\tilde{\boldsymbol{\eta}}(\mathbf{x}, t), \end{aligned} \quad (4.101)$$

where $\boldsymbol{w} := \mathbf{J}/(en)$. In this calculation we have made use of $\mathbf{v}(\mathbf{x}) + \alpha_s\mathbf{J}(\mathbf{x})/(en(\mathbf{x})) = \dot{\mathbf{q}}_{s0}(\mathbf{a}_s, t)\big|_{\mathbf{a}_s=\mathbf{q}_{s0}^{-1}(\mathbf{x},t)} = \mathbf{v}_s(\mathbf{x})$. We can also compute the variations of the Eulerian fields in terms of the Lagrangian displacements, which enables us to compare them with the dynamically accessible variations. Taking the first variation of (4.99) and identifying

$$\delta\dot{\mathbf{Q}} = \dot{\boldsymbol{\zeta}}\,, \qquad \delta\dot{\mathbf{D}} = \dot{\boldsymbol{\eta}}\,, \qquad \delta\mathbf{q}_s(\mathbf{a}_s,t)\big|_{\mathbf{a}_s=\mathbf{q}_s^{-1}(\mathbf{x},t)} = \tilde{\boldsymbol{\zeta}} + \alpha_s\tilde{\boldsymbol{\eta}}\,, \tag{4.102}$$

after some manipulations we find

$$\begin{aligned} \dot{\boldsymbol{\zeta}} &= \delta\mathbf{v} + \tilde{\boldsymbol{\zeta}}\cdot\nabla\mathbf{v} + \frac{m_i m_e}{m^2}\tilde{\boldsymbol{\eta}}\cdot\nabla\boldsymbol{w}\,, \\ \dot{\boldsymbol{\eta}} &= \delta\boldsymbol{w} + \tilde{\boldsymbol{\eta}}\cdot\nabla\mathbf{v} + \tilde{\boldsymbol{\zeta}}\cdot\nabla\boldsymbol{w} + \frac{m_e^2 - m_i^2}{m^2}\tilde{\boldsymbol{\eta}}\cdot\nabla\boldsymbol{w}\,. \end{aligned} \tag{4.103}$$

Next, combining Eqs. (4.101) with (4.103) the Eulerian variations of the fields $\mathbf{v}$, $\boldsymbol{w}$

$$\delta\mathbf{v} = \partial_t\tilde{\boldsymbol{\zeta}} + \mathbf{v}\cdot\nabla\tilde{\boldsymbol{\zeta}} - \tilde{\boldsymbol{\zeta}}\cdot\nabla\mathbf{v} + \frac{m_i m_e}{m^2}\left(\boldsymbol{w}\cdot\nabla\tilde{\boldsymbol{\eta}} - \tilde{\boldsymbol{\eta}}\cdot\nabla\boldsymbol{w}\right)\,, \tag{4.104}$$

$$\begin{aligned} \delta\boldsymbol{w} &= \partial_t\tilde{\boldsymbol{\eta}} + \mathbf{v}\cdot\nabla\tilde{\boldsymbol{\eta}} - \tilde{\boldsymbol{\eta}}\cdot\nabla\mathbf{v} + \boldsymbol{w}\cdot\nabla\tilde{\boldsymbol{\zeta}} - \tilde{\boldsymbol{\zeta}}\cdot\nabla\boldsymbol{w} \\ &\quad + \frac{m_e^2 - m_i^2}{m^2}\left(\boldsymbol{w}\cdot\nabla\tilde{\boldsymbol{\eta}} - \tilde{\boldsymbol{\eta}}\cdot\nabla\boldsymbol{w}\right)\,. \end{aligned} \tag{4.105}$$

Using the maps (4.99) and (4.101), and also relations (4.80) together with $d^3x = \mathcal{J}_s d^3\mathrm{a}_s$, we can compute the Eulerian expression for $\mathcal{L}_2$. Note that the role of the delta function in (4.87) is to ensure that $\mathbf{x} = \mathbf{x}'$, i.e. the trajectories $\mathbf{q}_i$ and $\mathbf{q}_e$ meet each other at $t > 0$. Upon inserting the inverse Lagrange-Euler maps (4.99)–(4.101) into the Lagrangian (4.98) we find

$$\begin{aligned} \mathcal{L}_2 = \; & \int d^3x \left\{ \frac{mn}{2}\left|\partial_t\boldsymbol{\zeta}\right|^2 + \frac{m_i m_e}{2m} n \left|\partial_t\boldsymbol{\eta}\right|^2 \right. \\ & + \partial_t\boldsymbol{\zeta}\cdot\left[mn\left(\mathbf{v}\cdot\nabla\boldsymbol{\zeta} + \frac{m_i m_e}{m^2}\boldsymbol{w}\cdot\nabla\boldsymbol{\eta}\right) + en\boldsymbol{\eta}\cdot\nabla\mathbf{A}\right] \\ & + \partial_t\boldsymbol{\eta}\cdot\left[\frac{m_i m_e}{m} n \left(\mathbf{v}\cdot\nabla\boldsymbol{\eta} + \boldsymbol{w}\cdot\nabla\boldsymbol{\zeta} + \frac{m_e^2 - m_i^2}{m^2}\boldsymbol{w}\cdot\nabla\boldsymbol{\eta}\right)\right. \\ & \left.\left. + en\left(\mathbf{A}_1 + \boldsymbol{\zeta}\cdot\nabla\mathbf{A} + \frac{m_e^2 - m_i^2}{m^2}\boldsymbol{\eta}\cdot\nabla\mathbf{A}\right)\right] + \mathfrak{W}(\boldsymbol{\zeta},\boldsymbol{\eta},\mathbf{A}_1,\Phi_1)\right\}, \end{aligned} \tag{4.106}$$

where

$$\begin{aligned} \mathfrak{W}(\boldsymbol{\zeta},\boldsymbol{\eta},\mathbf{A}_1,\Phi_1) = & -\tfrac{1}{2\mu_0}\left|\nabla\times\mathbf{A}_1\right|^2 + \tfrac{mn}{2}\left|\mathbf{v}\cdot\nabla\boldsymbol{\zeta} + \tfrac{m_i m_e}{m^2}\boldsymbol{w}\cdot\nabla\boldsymbol{\eta}\right|^2 \\ & + \tfrac{m_i m_e n}{2m}\left|\mathbf{v}\cdot\nabla\boldsymbol{\eta} + \boldsymbol{w}\cdot\nabla\boldsymbol{\zeta} + \tfrac{m_e^2 - m_i^2}{m^2}\boldsymbol{w}\cdot\nabla\boldsymbol{\eta}\right|^2 \\ & + en\left(\mathbf{v}\cdot\nabla\boldsymbol{\zeta} + \tfrac{m_i m_e}{m^2}\boldsymbol{w}\cdot\nabla\boldsymbol{\eta}\right)\cdot\left(\boldsymbol{\eta}\cdot\nabla\mathbf{A}\right) \\ & + en\left(\mathbf{v}\cdot\nabla\boldsymbol{\eta} + \boldsymbol{w}\cdot\nabla\boldsymbol{\zeta} + \tfrac{m_e^2-m_i^2}{m^2}\boldsymbol{w}\cdot\nabla\boldsymbol{\eta}\right)\cdot\left(\mathbf{A}_1 + \boldsymbol{\zeta}\cdot\nabla\mathbf{A} + \tfrac{m_e^2-m_i^2}{m^2}\boldsymbol{\eta}\cdot\nabla\mathbf{A}\right) \\ & + en\Big[\mathbf{v}\cdot\left(\boldsymbol{\eta}\cdot\nabla\mathbf{A}_1\right) + \boldsymbol{w}\cdot\left(\boldsymbol{\zeta}\cdot\nabla\mathbf{A}_1\right) + \mathbf{v}\cdot\left(\boldsymbol{\zeta}\boldsymbol{\eta}:\nabla\nabla\mathbf{A}\right) + \tfrac{1}{2}\boldsymbol{w}\cdot\left(\boldsymbol{\zeta}\boldsymbol{\zeta}:\nabla\nabla\mathbf{A}\right) \\ & + \tfrac{m_e^2-m_i^2}{m^2}\boldsymbol{w}\cdot\left(\boldsymbol{\eta}\cdot\nabla\mathbf{A}_1\right) + \tfrac{m_e^2-m_i^2}{m^2}\boldsymbol{w}\cdot\left(\boldsymbol{\zeta}\boldsymbol{\eta}:\nabla\nabla\mathbf{A}\right) + \tfrac{m_e^2-m_i^2}{2m^2}\mathbf{v}\cdot\left(\boldsymbol{\eta}\boldsymbol{\eta}:\nabla\nabla\mathbf{A}\right) \end{aligned}$$

$$
+\tfrac{m_e^3+m_i^3}{2m^3}\boldsymbol{w}\cdot(\boldsymbol{\eta}\boldsymbol{\eta}:\nabla\nabla\mathbf{A})-\boldsymbol{\zeta}\boldsymbol{\eta}:\nabla\nabla\Phi-\tfrac{m_e^2-m_i^2}{2m^2}\boldsymbol{\eta}\boldsymbol{\eta}:\nabla\nabla\Phi-\boldsymbol{\eta}\cdot\nabla\Phi_1\Big]
$$

$$
-\tfrac{p}{2}\left[\nabla\boldsymbol{\zeta}:\nabla\boldsymbol{\zeta}-(\nabla\cdot\boldsymbol{\zeta})^2\right]-\tfrac{1}{2}n\tfrac{\partial p}{\partial n}(\nabla\cdot\boldsymbol{\zeta})^2
$$

$$
-\left[\nabla\boldsymbol{\zeta}:\nabla\boldsymbol{\eta}-(\nabla\cdot\boldsymbol{\zeta})(\nabla\cdot\boldsymbol{\eta})\right]\left(\tfrac{m_e}{m}p_i-\tfrac{m_i}{m}p_e\right)-n\left(\tfrac{m_e}{m}\tfrac{\partial p_i}{\partial n}-\tfrac{m_i}{m}\tfrac{\partial p_e}{\partial n}\right)(\nabla\cdot\boldsymbol{\zeta})(\nabla\cdot\boldsymbol{\eta})
$$

$$
-\tfrac{1}{2}\left[\nabla\boldsymbol{\eta}:\nabla\boldsymbol{\eta}-(\nabla\cdot\boldsymbol{\eta})^2\right]\left[\left(\tfrac{m_e}{m}\right)^2p_i+\left(\tfrac{m_i}{m}\right)^2p_e\right]
$$

$$
-\tfrac{1}{2}n\left[\left(\tfrac{m_e}{m}\right)^2\tfrac{\partial p_i}{\partial n}+\left(\tfrac{m_i}{m}\right)^2\tfrac{\partial p_e}{\partial n}\right](\nabla\cdot\boldsymbol{\eta})^2\,. \tag{4.107}
$$

Here we have used $p_s = n^2\partial\mathcal{U}_s/\partial n$, the Dalton's law $p = p_i + p_e$ and in addition $n^3\partial^2\mathcal{U}_s/\partial n^2 = n\partial p_s/\partial n - 2p_s$. The tildes have been dropped since we are working now in a completely Eulerian framework and there is no need to distinguish from the Lagrangian displacements. We should stress here that the XMHD model we use in the previous sections was derived upon expanding the quasineutral two-fluid equations and keeping terms up to zeroth order in $\mu := m_e/m_i$ in the Alfvén normalized equations of motion. In this section however we have not performed such an expansion and therefore up to now the results are fully two-fluid with quasineutrality. Hence, they can be used either to describe an ion-electron plasma or a positron-electron plasma, just by replacing the ion mass by the positron mass.

The Euler-Lagrange equations that correspond to (4.106) are obtained upon minimizing the action

$$
\mathcal{S}_2=\int_{t_1}^{t_2}dt\mathcal{L}_2\,, \tag{4.108}
$$

with boundary conditions $\boldsymbol{\zeta}\cdot\hat{\mathrm{n}}=\boldsymbol{\eta}\cdot\hat{\mathrm{n}}=0$, where $\hat{\mathrm{n}}$ is the unit vector normal to the boundary, and $\boldsymbol{\zeta}(\mathbf{x},t=t_1)=\boldsymbol{\zeta}(\mathbf{x},t=t_2)=\boldsymbol{\eta}(\mathbf{x},t=t_1)=\boldsymbol{\eta}(\mathbf{x},t=t_2)=0$. These equations describe the linearized dynamics; more specifically, from the $\boldsymbol{\zeta}$-variation one would take the linearized momentum equation while from $\boldsymbol{\eta}$-variations a generalized Ohm's law occurs. However, there are two redundant variables, namely $\mathbf{A}_1$ and Φ_1, which do not appear in pairs of generalized coordinates and velocities. In some way we need to express them in terms of the generalized coordinates so as to eliminate this redundancy. As regards Φ_1 one can express it by selecting a particular gauge. Alternatively we can minimize the action with respect to these variables only and find the respective "Euler-Lagrange equations" which can be used either to eliminate Φ_1 and $\mathbf{A}_1$ or as side conditions. Accordingly, minimizing the action with respect to the electromagnetic field variables we find

$$
\delta\Phi_1\ :\ e\nabla\cdot(n\boldsymbol{\eta})=0\,, \tag{4.109}
$$

$$
\delta\mathbf{A}_1\ :\ en\Big[\partial_t\boldsymbol{\eta}+\mathbf{v}\cdot\nabla\boldsymbol{\eta}-\boldsymbol{\eta}\cdot\nabla\mathbf{v}+\boldsymbol{w}\cdot\nabla\boldsymbol{\zeta}-\boldsymbol{\zeta}\cdot\nabla\boldsymbol{w}
+\frac{m_e^2-m_i^2}{m^2}(\boldsymbol{w}\cdot\nabla\boldsymbol{\eta}-\boldsymbol{\eta}\cdot\nabla\boldsymbol{w})\Big]-\frac{\mathbf{J}}{n}\nabla\cdot(n\boldsymbol{\zeta})-\mathbf{J}_1=0\,, \tag{4.110}
$$

where for the derivation of (4.110) we need to assume $(\mathbf{A}_1\times\delta\mathbf{A}_1)\big|_{\partial D}\cdot\hat{\mathrm{n}}=0$. Equation (4.109) expresses charge neutrality for the perturbed state and is also a manifestation of gauge invariance of Lagrangian (4.106). In view of this condition the term that contains Φ_1 in $\mathfrak{W}$ can be eliminated upon integrating by parts. Also, combining Eq. (4.110) with (4.105) we find the expression for the Eulerian variation of the particle density to be

$$n_1=-\nabla\cdot(n\boldsymbol{\zeta})\,, \tag{4.111}$$

which is of the form of the dynamically accessible variation $\delta\rho_{da}$ (see Eq. (4.61)).

To arrive at a sufficient stability condition we need to calculate the Hamiltonian of the linearized dynamics. To this end the standard procedure of Legendre transforming the Lagrangian (4.106) can be applied. The departing point for performing this transformation is to define the generalized momenta $\boldsymbol{\pi}_\zeta$, $\boldsymbol{\pi}_\eta$ as follows

$$\boldsymbol{\pi}_\zeta:=\frac{\delta\mathcal{L}_2}{\delta\dot{\boldsymbol{\zeta}}}=mn\left(\partial_t\boldsymbol{\zeta}+\mathbf{v}\cdot\nabla\boldsymbol{\zeta}+\frac{m_im_e}{m^2}\boldsymbol{w}\cdot\nabla\boldsymbol{\eta}\right)+en\boldsymbol{\eta}\cdot\nabla\mathbf{A}\,, \tag{4.112}$$

$$\begin{aligned}\boldsymbol{\pi}_\eta:=\frac{\delta\mathcal{L}_2}{\delta\dot{\boldsymbol{\eta}}}=\frac{m_im_e}{m}n&\left(\partial_t\boldsymbol{\eta}+\mathbf{v}\cdot\nabla\boldsymbol{\eta}+\boldsymbol{w}\cdot\nabla\boldsymbol{\zeta}+\frac{m_e^2-m_i^2}{m^2}\boldsymbol{w}\cdot\nabla\boldsymbol{\eta}\right)\\ &+en\left(\mathbf{A}_1+\boldsymbol{\zeta}\cdot\nabla\mathbf{A}+\frac{m_e^2-m_i^2}{m^2}\boldsymbol{\eta}\cdot\nabla\mathbf{A}\right)\,.\end{aligned} \tag{4.113}$$

With Eqs. (4.112)–(4.113) at hand, employing the usual Legendre transform

$$\mathcal{H}_2=\int_D d^3x\;(\boldsymbol{\pi}_\zeta\cdot\partial_t\boldsymbol{\zeta}+\boldsymbol{\pi}_\eta\cdot\partial_t\boldsymbol{\eta})-\mathcal{L}_2\,,$$

we find

$$\begin{aligned}\mathcal{H}_2=\int_D d^3x&\left[\frac{1}{2mn}\left|\boldsymbol{\pi}_\zeta-mn\left(\mathbf{v}\cdot\nabla\boldsymbol{\zeta}+\frac{m_im_e}{m^2}\boldsymbol{w}\cdot\nabla\boldsymbol{\eta}\right)-en\boldsymbol{\eta}\cdot\nabla\mathbf{A}\right|^2\right.\\ &+\frac{m}{2m_im_en}\left|\boldsymbol{\pi}_\eta-\frac{m_im_e}{m}n\left(\mathbf{v}\cdot\nabla\boldsymbol{\eta}+\boldsymbol{w}\cdot\nabla\boldsymbol{\zeta}+\frac{m_e^2-m_i^2}{m^2}\boldsymbol{w}\cdot\nabla\boldsymbol{\eta}\right)\right.\\ &\left.\left.-en\left(\mathbf{A}_1+\boldsymbol{\zeta}\cdot\nabla\mathbf{A}+\frac{m_e^2-m_i^2}{m^2}\boldsymbol{\eta}\cdot\nabla\mathbf{A}\right)\right|^2-\mathfrak{W}(\boldsymbol{\zeta},\boldsymbol{\eta})\right]\,.\end{aligned} \tag{4.114}$$

From (4.114) we deduce that

$$-\int d^3x\,\mathfrak{W}(\boldsymbol{\zeta},\boldsymbol{\eta})\geq 0 \tag{4.115}$$

with $\mathfrak{W}(\boldsymbol{\zeta},\boldsymbol{\eta})$ given by (4.107) implies stability.

4.5 Hall MHD

The HMHD case has an interesting peculiarity: to derive the HMHD perturbed Lagrangian we assume massless electrons i.e. $m_e = 0$, as a result, $\partial_t\boldsymbol{\eta}$ appears linearly in $\mathcal{L}_2$ and therefore the definition of the canonical momentum $\boldsymbol{\pi}_\eta$ results in a constraint instead of an equation that can be used to express $\partial_t\boldsymbol{\eta}$ in terms of $\boldsymbol{\pi}_\eta$. But before addressing this peculiarity we Alfvén normalize the HMHD Lagrangian, term by term, so as to facilitate the comparisons with already known results in this framework. The Alfvén normalization is effected by

$$\begin{aligned}
&\bar{n} = n/n_0\,, \quad \bar{t} = t/\tau_A\,, \quad \bar{\mathbf{B}} = \mathbf{B}/B_0\,,\\
&\bar{\mathbf{J}} = \mathbf{J}/(B_0/\ell\mu_0)\,, \quad \bar{\nabla} = \ell\nabla\,, \quad \bar{\mathbf{A}} = \mathbf{A}/(\ell B_0)\,,\\
&\bar{\mathbf{E}} = \mathbf{E}/(v_A B_0)\,, \quad \bar{\Phi} = \Phi/(\ell v_A B_0)\,, \quad \bar{p_s} = p_s/(B_0^2/\mu_0)\,,
\end{aligned} \tag{4.116}$$

where ℓ, n_0 and B_0 are reference length, particle density and magnetic field, respectively; $v_A = B_0/\sqrt{\mu_0 m_i n_0}$ is the Alfvén speed and $\tau_A = \ell/v_A$ is the Alfvén time. In order to write the Lagrangian in dimensionless form we need also to introduce normalized displacements $\boldsymbol{\zeta}$ and $\boldsymbol{\eta}$. Equations (4.104) and (4.105) suggest that an appropriate normalization is

$$\bar{\boldsymbol{\zeta}} = \boldsymbol{\zeta}/\ell\,, \quad \bar{\boldsymbol{\eta}} = \boldsymbol{\eta}/\sqrt{m_i/\mu_0 n_0 e^2} = \boldsymbol{\eta}/\lambda_i\,, \tag{4.117}$$

where λ_i is the ion skin depth ($\lambda_i = d_i\ell$). In view of (4.116) and (4.117) and setting $m_e = 0$ Lagrangian (4.106) can be brought in the following dimensionless form

$$\begin{aligned}
\mathcal{L}_2 = \quad & \int d^3x \left\{ \frac{\rho}{2}\left|\partial_t\boldsymbol{\zeta}\right|^2 + \rho\,(\partial_t\boldsymbol{\zeta})\cdot(\boldsymbol{\eta}\cdot\nabla\mathbf{A} + \mathbf{v}\cdot\nabla\boldsymbol{\zeta}) \right.\\
& \left. +\rho(\partial_t\boldsymbol{\eta})\cdot(\mathbf{A}_1 + \boldsymbol{\zeta}\cdot\nabla\mathbf{A} - d_i\boldsymbol{\eta}\cdot\nabla\mathbf{A}) + \mathfrak{W}_{hmhd}(\boldsymbol{\zeta},\boldsymbol{\eta}) \right\},
\end{aligned} \tag{4.118}$$

where

$$\begin{aligned}
\mathfrak{W}_{hmhd} = \quad & \frac{\rho}{2}\left|\mathbf{v}\cdot\nabla\boldsymbol{\zeta}\right|^2 + \rho(\mathbf{v}\cdot\nabla\boldsymbol{\zeta})\cdot(\boldsymbol{\eta}\cdot\nabla\mathbf{A})\\
& +\rho\left(\mathbf{v}\cdot\nabla\boldsymbol{\eta} + \rho^{-1}\mathbf{J}\cdot\nabla\boldsymbol{\zeta} - d_i\rho^{-1}\mathbf{J}\cdot\nabla\boldsymbol{\eta}\right)\cdot(\mathbf{A}_1 + \boldsymbol{\zeta}\cdot\nabla\mathbf{A} - d_i\boldsymbol{\eta}\cdot\nabla\mathbf{A})\\
& +\rho\mathbf{v}\cdot(\boldsymbol{\eta}\cdot\nabla\mathbf{A}_1) + \rho\mathbf{v}\cdot(\boldsymbol{\zeta}\boldsymbol{\eta}:\nabla\nabla\mathbf{A}) - \frac{d_i}{2}\rho\mathbf{v}\cdot(\boldsymbol{\eta}\boldsymbol{\eta}:\nabla\nabla\mathbf{A})\\
& +\mathbf{J}\cdot(\boldsymbol{\zeta}\cdot\nabla\mathbf{A}_1) - d_i\mathbf{J}\cdot(\boldsymbol{\eta}\cdot\nabla\mathbf{A}_1) + \frac{1}{2}\mathbf{J}\cdot(\boldsymbol{\zeta}\boldsymbol{\zeta}:\nabla\nabla\mathbf{A})\\
& -d_i\mathbf{J}\cdot(\boldsymbol{\zeta}\boldsymbol{\eta}:\nabla\nabla\mathbf{A}) + \frac{d_i^2}{2}\mathbf{J}\cdot(\boldsymbol{\eta}\boldsymbol{\eta}:\nabla\nabla\mathbf{A}) - \rho\boldsymbol{\eta}\cdot\nabla\Phi_1\\
& -\rho(\boldsymbol{\zeta}\boldsymbol{\eta}:\nabla\nabla\Phi) + \frac{d_i}{2}\rho\boldsymbol{\eta}\boldsymbol{\eta}:\nabla\nabla\Phi - \frac{p}{2}[\nabla\boldsymbol{\zeta}:\nabla\boldsymbol{\zeta} - (\nabla\cdot\boldsymbol{\zeta})^2]\\
& -\frac{\rho}{2}c_s^2(\nabla\cdot\boldsymbol{\zeta})^2 + d_i p_e[\nabla\boldsymbol{\zeta}:\nabla\boldsymbol{\eta} - (\nabla\cdot\boldsymbol{\zeta})(\nabla\cdot\boldsymbol{\eta})] + d_i\frac{\rho}{2}c_{se}^2(\nabla\cdot\boldsymbol{\zeta})(\nabla\cdot\boldsymbol{\eta})
\end{aligned}$$

$$-\frac{d_i^2}{2}p_e[\nabla\boldsymbol{\eta} : \nabla\boldsymbol{\eta} - (\nabla\cdot\boldsymbol{\eta})^2] - \frac{d_i^2}{2}\rho c_{se}^2(\nabla\cdot\boldsymbol{\eta})^2 - \frac{1}{2}|\mathbf{B}_1|^2\Big\}\,, \tag{4.119}$$

and the bars have been dropped. In addition, the perturbation of the velocity field and of the field $\mathbf{J}/\rho$ are given by

$$\delta\mathbf{v} = \partial_t\boldsymbol{\zeta} + \mathbf{v}\cdot\nabla\boldsymbol{\zeta} - \boldsymbol{\zeta}\cdot\nabla\mathbf{v}\,, \tag{4.120}$$

$$\begin{aligned}\delta\left(\frac{\mathbf{J}}{\rho}\right) &= \partial_t\boldsymbol{\eta} + \mathbf{v}\cdot\nabla\boldsymbol{\eta} - \boldsymbol{\eta}\cdot\nabla\mathbf{v} + \frac{\mathbf{J}}{\rho}\cdot\nabla\boldsymbol{\zeta} - \boldsymbol{\zeta}\cdot\nabla\frac{\mathbf{J}}{\rho} \\ &\quad - d_i\left(\frac{\mathbf{J}}{\rho}\cdot\nabla\boldsymbol{\eta} - \boldsymbol{\eta}\cdot\nabla\frac{\mathbf{J}}{\rho}\right)\,,\end{aligned} \tag{4.121}$$

while the generalized momenta $\boldsymbol{\pi}_{\boldsymbol{\zeta}}$ and $\boldsymbol{\pi}_{\boldsymbol{\eta}}$ are now computed as follows

$$\boldsymbol{\pi}_\zeta = \frac{\delta\mathcal{L}_2}{\delta\dot{\boldsymbol{\zeta}}} = \rho(\partial_t\boldsymbol{\zeta} + \mathbf{v}\cdot\nabla\boldsymbol{\zeta}) + \rho\boldsymbol{\eta}\cdot\nabla\mathbf{A}\,, \tag{4.122}$$

$$\boldsymbol{\pi}_\eta = \frac{\delta\mathcal{L}_2}{\delta\dot{\boldsymbol{\eta}}} = \rho\left(\mathbf{A}_1 + \boldsymbol{\zeta}\cdot\nabla\mathbf{A} - d_i\boldsymbol{\eta}\cdot\nabla\mathbf{A}\right)\,, \tag{4.123}$$

Note that Eq. (4.123) cannot be used in order to express $\partial_t\boldsymbol{\eta}$ in terms of $\boldsymbol{\pi}_\eta$; therefore it can be interpreted as a constraint between the dynamical variables, which helps us though to express explicitly $\mathbf{A}_1$ in terms of canonical variables via $\mathbf{A}_1 = \rho^{-1}\boldsymbol{\pi}_\eta - (\boldsymbol{\zeta} - d_i\boldsymbol{\eta})\cdot\nabla\mathbf{A}$. A consistency condition is that this equation holds for all time i.e. that it is preserved by the dynamics

$$[\boldsymbol{\pi}_\eta - \rho\left(\mathbf{A}_1 + \boldsymbol{\zeta}\cdot\nabla\mathbf{A} - d_i\boldsymbol{\eta}\cdot\nabla\mathbf{A}\right), \mathcal{H}_2] = 0\,, \tag{4.124}$$

where

$$[f,g] = \int d^3x\left(\frac{\delta f}{\delta\boldsymbol{\zeta}}\cdot\frac{\delta g}{\delta\boldsymbol{\pi}_\zeta} - \frac{\delta g}{\delta\boldsymbol{\zeta}}\cdot\frac{\delta f}{\delta\boldsymbol{\pi}_\zeta} + \frac{\delta f}{\delta\boldsymbol{\eta}}\cdot\frac{\delta g}{\delta\boldsymbol{\pi}_\eta} - \frac{\delta g}{\delta\boldsymbol{\eta}}\cdot\frac{\delta f}{\delta\boldsymbol{\pi}_\eta}\right) \tag{4.125}$$

is the canonical Poisson bracket and

$$\begin{aligned}\mathcal{H}_2 &= \int d^3x(\boldsymbol{\pi}_\zeta\cdot\partial_t\boldsymbol{\zeta} + \boldsymbol{\pi}_\eta\cdot\partial_t\boldsymbol{\eta}) - \mathcal{L}_2 \\ &= \int d^3x\left[\frac{1}{2\rho}|\boldsymbol{\pi}_\zeta - \rho\mathbf{v}\cdot\nabla\boldsymbol{\zeta} - \rho\boldsymbol{\eta}\cdot\nabla\mathbf{A}|^2 - \mathfrak{W}_{hmhd}(\boldsymbol{\zeta},\boldsymbol{\eta},\boldsymbol{\pi}_\eta)\right],\end{aligned} \tag{4.126}$$

where $\mathbf{A}_1$ has been expressed via Eq. (4.123). From (4.124) (4.125) and (4.126) we find

$$\begin{aligned}-\frac{\partial\mathfrak{W}_{hmhd}}{\partial\boldsymbol{\eta}} &= d_i\nabla\mathbf{A}\cdot\Big\{\frac{\mathbf{J}}{\rho}\nabla\cdot(\rho\boldsymbol{\zeta}) + \nabla\times\nabla\times\left[\rho^{-1}\boldsymbol{\pi}_\eta - (\boldsymbol{\zeta} - d_i\boldsymbol{\eta})\cdot\nabla\mathbf{A}\right] \\ &\quad + \rho\left[(\boldsymbol{\zeta} - d_i\boldsymbol{\eta})\cdot\nabla\frac{\mathbf{J}}{\rho} - \mathbf{v}\cdot\nabla\boldsymbol{\eta} + \boldsymbol{\eta}\cdot\nabla\mathbf{v} - \frac{\mathbf{J}}{\rho}\cdot\nabla\boldsymbol{\zeta} + d_i\frac{\mathbf{J}}{\rho}\cdot\nabla\boldsymbol{\eta}\right]\Big\}\,.\end{aligned} \tag{4.127}$$

Now let us proceed by computing Hamilton's equations of motion

$$\begin{aligned}
\partial_t \boldsymbol{\eta} = & \frac{\delta \mathcal{H}_2}{\delta \boldsymbol{\pi}_\eta} = -\mathbf{v}\cdot\nabla\boldsymbol{\eta} + \boldsymbol{\eta}\cdot\nabla\mathbf{v} - \frac{\mathbf{J}}{\rho}\cdot\nabla\boldsymbol{\zeta} + \boldsymbol{\zeta}\cdot\nabla\frac{\mathbf{J}}{\rho} + \frac{\mathbf{J}}{\rho^2}\nabla\cdot(\rho\boldsymbol{\zeta}) \\
& + d_i\left(\frac{\mathbf{J}}{\rho}\cdot\nabla\boldsymbol{\eta} - \boldsymbol{\eta}\cdot\nabla\frac{\mathbf{J}}{\rho}\right) + \rho^{-1}\nabla\times\nabla\times\left[\rho^{-1}\boldsymbol{\pi}_\eta - (\boldsymbol{\zeta} - d_i\boldsymbol{\eta})\cdot\nabla\mathbf{A}\right], \quad (4.128) \\
\partial_t \boldsymbol{\zeta} = & \frac{\delta \mathcal{H}_2}{\delta \boldsymbol{\pi}_\zeta} = \rho^{-1}(\boldsymbol{\pi}_\zeta - \rho\mathbf{v}\cdot\nabla\boldsymbol{\zeta} - \rho\boldsymbol{\eta}\cdot\nabla\mathbf{A}), \quad (4.129) \\
\partial_t \boldsymbol{\pi}_\eta = & -\frac{\delta \mathcal{H}_2}{\delta \boldsymbol{\eta}} = \nabla\mathbf{A}\cdot(\boldsymbol{\pi}_\zeta - \rho\mathbf{v}\cdot\nabla\boldsymbol{\zeta} - \rho\boldsymbol{\eta}\cdot\nabla\mathbf{A}) - \frac{\partial \mathfrak{W}_{hmhd}}{\partial\boldsymbol{\eta}}, \quad (4.130) \\
\partial_t \boldsymbol{\pi}_\zeta = & -\frac{\delta \mathcal{H}_2}{\delta \boldsymbol{\zeta}} = -\Big\{\rho\mathbf{v}\cdot\nabla[\rho^{-1}\boldsymbol{\pi}_\zeta - \mathbf{v}\cdot\nabla\boldsymbol{\zeta} - \boldsymbol{\eta}\cdot\nabla\mathbf{A}] + \rho\mathbf{v}\cdot\nabla(\boldsymbol{\eta}\cdot\nabla\mathbf{A}) \\
& + \rho\mathbf{v}\cdot\nabla(\mathbf{v}\cdot\nabla\boldsymbol{\zeta}) + \mathbf{J}\cdot\nabla\frac{\boldsymbol{\pi}_\eta}{\rho} - \rho\nabla\mathbf{A}\cdot\Big[(\boldsymbol{\eta}\cdot\nabla\mathbf{v}) + (\boldsymbol{\zeta} - d_i\boldsymbol{\eta})\cdot\nabla\frac{\mathbf{J}}{\rho} \\
& + \frac{\mathbf{J}}{\rho}\nabla\cdot(\rho\boldsymbol{\zeta}) + \nabla\times\nabla\times(\rho^{-1}\boldsymbol{\pi}_\eta - \boldsymbol{\zeta}\cdot\nabla\mathbf{A} + d_i\boldsymbol{\eta}\cdot\nabla\mathbf{A})\Big] \\
& - \rho(\boldsymbol{\eta}\cdot\nabla\nabla\mathbf{A})\cdot\mathbf{v} - \nabla\left[\rho^{-1}\boldsymbol{\pi}_\eta - (\boldsymbol{\zeta} - d_i\boldsymbol{\eta})\cdot\nabla\mathbf{A}\right]\cdot\mathbf{J} - (\boldsymbol{\zeta}\cdot\nabla\nabla\mathbf{A})\cdot\mathbf{J} \\
& + d_i(\boldsymbol{\eta}\cdot\nabla\nabla\mathbf{A})\cdot\mathbf{J} + \rho\boldsymbol{\eta}\cdot\nabla\nabla\Phi + \nabla p\nabla\cdot\boldsymbol{\zeta} - \nabla\boldsymbol{\zeta}\cdot\nabla p - \nabla\left(\rho\frac{\partial p}{\partial\rho}\nabla\cdot\boldsymbol{\zeta}\right) \\
& - d_i\nabla p_e\nabla\cdot\boldsymbol{\eta} + d_i\nabla\boldsymbol{\eta}\cdot\nabla p_e + d_i\nabla\left(\rho\frac{\partial p_e}{\partial\rho}\nabla\cdot\boldsymbol{\eta}\right)\Big\}. \quad (4.131)
\end{aligned}$$

Combining (4.128) with (4.123) and (4.121) gives

$$\rho_1 = -\nabla\cdot(\rho\boldsymbol{\zeta}). \quad (4.132)$$

Equation (4.129) is merely the definition of the canonical momentum $\boldsymbol{\pi}_\zeta$. Exploiting (4.122), (4.123) and relations (4.120) and (4.121) and also the stationary momentum equation and Ohm's law, which are given by

$$\mathbf{v}\cdot\nabla\mathbf{v} - \rho^{-1}\mathbf{J}\times\mathbf{B} + \rho^{-1}\nabla p = 0, \quad (4.133)$$

$$-\nabla\Phi + \left(\mathbf{v} - d_i\frac{\mathbf{J}}{\rho}\right)\times\mathbf{B} + \rho^{-1}\nabla p_e = 0, \quad (4.134)$$

we can corroborate that (4.130) and (4.131) give the perturbed Ohm's law and momentum equation, respectively. Therefore, Hamiltonian (4.126) describes correctly the linearized HMHD dynamics. Note that $\mathfrak{W}_{hmhd}$ is not yet fully expressed in terms of the displacement vectors $\boldsymbol{\zeta}, \boldsymbol{\eta}$ due to $\boldsymbol{\pi}_\eta$ (or $\mathbf{A}_1$ through (4.123)) which appears explicitly in its expression. We can overcome this by combining the consistency condition (4.127) with Hamilton's equations (4.128) and (4.130) to find

$$\partial_t\mathbf{A}_1 = \partial_t(\boldsymbol{\zeta} - d_i\boldsymbol{\eta})\times\mathbf{B}_0. \quad (4.135)$$

Integration of (4.135) introduces in general a time independent vector, which however should vanish because otherwise time independent terms would appear in the perturbed equations of motion. Therefore $\mathbf{A}_1 = (\boldsymbol{\zeta} - d_i\boldsymbol{\eta})\times\mathbf{B}_0$ or $\mathbf{B}_1 = \nabla\times[(\boldsymbol{\zeta} - d_i\boldsymbol{\eta})\times\mathbf{B}_0]$ which is well known solution of the perturbed induction equation (see [15]). This expression is similar with the corresponding expression in ideal MHD. The difference is the appearance of the displacement vector $\boldsymbol{\eta}$ multiplied by d_i so the MHD result can be recovered for $d_i \to 0$. This is an anticipated result, since the fluid velocity in the MHD induction equation is replaced by $\mathbf{v} - d_i\mathbf{J}$ in the HMHD case. After this analysis we conclude that

$$-\int d^3x\, \mathfrak{W}_{hmhd}(\boldsymbol{\zeta}, \boldsymbol{\eta}) \geq 0\,, \tag{4.136}$$

where $\mathfrak{W}_{hmhd}(\boldsymbol{\zeta}, \boldsymbol{\eta})$ is given by (4.119) with $\mathbf{A}_1 = (\boldsymbol{\zeta} - d_i\boldsymbol{\eta}) \times \mathbf{B}_0$, is sufficient for stability. Note that the term containing $\nabla\Phi_1$ can be neglected in view of $\nabla \cdot (\rho\boldsymbol{\eta}) = 0$ and $\boldsymbol{\eta} \cdot \hat{\mathrm{n}}\big|_{\partial D} = 0$.

Chapter 5

Alternative bracket formulations for tsXMHD and RMHD

In this chapter, alternative brackets describing the dynamics of ideal incompressible XMHD, translationally symmetric XMHD (tsXMHD) and reduced MHD (RMHD), are presented. Also, we formulate a heuristic method to construct the dynamical equations of 2D fluid models from the conservation of the corresponding Hamiltonian and Casimir invariants. The findings of this second part are published in [119].

The above concepts are elaborated in two sections: in Section 5.1 a trilinear bracket formulation and also an alternative bilinear bracket for XMHD and tsXMHD are found while in Section 5.2 we present the inverse approach mentioned above.

5.1 Alternative bracket formulations for XMHD and tsXMHD

It is known that the equations of hydrodynamics and magnetohydrodynamics can be cast in a generalized Hamiltonian form in terms of trilinear, instead of bilinear brackets (see [120] and [121]), called Nambu brackets. It should be stressed though that the Nambu brackets discussed herein and in the references [120, 121] are different from the finite dimensional brackets introduced in the classic paper of Nambu [122], since they are infinite dimensional generalizations of the latter that may or may not not satisfy generalized Jacobi identities for Nambu brackets e.g. [123]. Such an infinite dimensional generalization of the Nambu bracket using Lie algebraic considerations was introduced initially in [124] and applied to the Weyl-Wigner formalism of quantum mechanics. Later on, similar formalisms emerged for fluid dynamics where they proved to be practically useful for performing conservative numerical integration because they are fully antisymmetric. Namely one can construct conservative algorithms in the context of 2D hydrodynamics [125, 126], exploiting the fact that the Hamiltonian and the Casimir invariant (Enstrophy) are conserved up to machine precision if it is ensured that the discretization procedure retains the antisymmetry property. Hence, it is of interest to derive such brackets as a step towards the construction of analogous Casimir preserving algorithms for plasma dynamics. In addition, once the Nambu bracket is formulated one can derive an alternative bilinear bracket upon substituting the original Hamiltonian, which is now embedded into the very structure of the resulting bracket, just as the Casimir invariant was embedded into the structure of the original Poisson bracket. The new Hamiltonian is the Casimir invariant which was initially used for the construction of the Nambu structure. However, proof of the validity or not of this new bracket's Jacobi identity is not pursued.

In the following analysis we adopt the convenient vorticity representation of incompressible XMHD dynamics obtained upon acting on the momentum equation with the curl operator

$$\partial_t\boldsymbol{\omega} = \nabla\times(\mathbf{v}\times\boldsymbol{\omega}+\mathbf{J}\times\mathbf{B}^*)\,, \tag{5.1}$$

$$\partial_t\mathbf{B}^* = \nabla\times\big(\mathbf{v}\times\mathbf{B}^*-d_i\mathbf{J}\times\mathbf{B}^*+d_e^2\mathbf{J}\times\boldsymbol{\omega}\big)\,, \tag{5.2}$$

where $\boldsymbol{\omega}=\nabla\times\mathbf{v}$. The Poisson bracket in vorticity representation takes the following form

$$\begin{aligned}\{F,G\} = \int d^3x \Big\{&(\nabla\times\mathbf{v})\cdot[(\nabla\times F_{\boldsymbol{\omega}})\times(\nabla\times G_{\boldsymbol{\omega}})]\\ &\mathbf{B}^*\cdot[(\nabla\times F_{\boldsymbol{\omega}})\times(\nabla\times G_{\mathbf{B}^*})-(\nabla\times G_{\boldsymbol{\omega}})\times(\nabla\times F_{\mathbf{B}^*})]\\ &-d_i\mathbf{B}^*\cdot[(\nabla\times F_{\mathbf{B}^*})\times(\nabla\times G_{\mathbf{B}^*})]\\ &+d_e^2(\nabla\times\mathbf{v})\cdot[(\nabla\times F_{\mathbf{B}^*})\times(\nabla\times G_{\mathbf{B}^*})]\Big\}\,.\end{aligned} \tag{5.3}$$

5.1.1 Nambu bracket formulation

To find a trilinear bracket, equivalent to (5.3) we need somehow to incorporate a Casimir invariant. Adding the Casimirs in (1.74) we obtain

$$\mathcal{C}_+ + \mathcal{C}_- = \int d^3x\left(\mathbf{A}^*\cdot\mathbf{B}^*+d_i\mathbf{v}\cdot\mathbf{B}^*+\frac{d_i^2}{2}\mathbf{v}\cdot\boldsymbol{\omega}+d_e^2\mathbf{v}\cdot\boldsymbol{\omega}\right)\,, \tag{5.4}$$

while taking their difference yields

$$\mathcal{C}_+ - \mathcal{C}_- = \sqrt{d_i^2+4d_e^2}\int d^3x\left(\mathbf{v}\cdot\mathbf{B}^*+\frac{d_i}{2}\mathbf{v}\cdot\boldsymbol{\omega}\right)\,. \tag{5.5}$$

Combining these two results we find that

$$\mathcal{C} = \frac{1}{2}\int d^3x\,\left(\mathbf{A}^*\cdot\mathbf{B}^*+d_e^2\mathbf{v}\cdot\boldsymbol{\omega}\right)\,, \tag{5.6}$$

is also a Casimir. Having this alternative Casimir, it is not difficult to identify that (5.3) can be written as

$$\begin{aligned}\{F,G,\mathcal{C}\} = \int d^3x \Big\{ &\frac{1}{3d_e^2}(\nabla\times F_{\boldsymbol{\omega}})\cdot[(\nabla\times G_{\boldsymbol{\omega}})\times(\nabla\times\mathcal{C}_{\boldsymbol{\omega}})] \\ &+(\nabla\times F_{\mathbf{B}}^{*})\cdot[(\nabla\times G_{\boldsymbol{\omega}})\times(\nabla\times\mathcal{C}_{\mathbf{B}^*})] \\ -\frac{d_i}{3}(\nabla\times F_{\mathbf{B}}^{*})\cdot[(\nabla\times G_{\mathbf{B}^*})&\times(\nabla\times\mathcal{C}_{\mathbf{B}^*})]\Big\} + \circlearrowleft(F,G,\mathcal{C})\,, \end{aligned} \tag{5.7}$$

where $\circlearrowleft$ denotes cyclic permutation. In view of (5.7) the dynamics is correctly described by the following generalized Hamilton's equation

$$\partial_t f = \{f,\mathcal{H},\mathcal{C}\}\,, \tag{5.8}$$

where f is an arbitrary functional. It is easy to corroborate that for $f(\mathbf{x}') = \int d^3x\boldsymbol{\omega}(\mathbf{x})\delta(\mathbf{x}-\mathbf{x}')$ and $f(\mathbf{x}') = \int d^3x\mathbf{B}^*(\mathbf{x})\delta(\mathbf{x}-\mathbf{x}')$ we retrieve Eqs. (5.2).

5.1.2 Alternative bilinear form

The form (5.7), which designates that $\mathcal{H}$ and $\mathcal{C}$ can be interchanged and in addition is immediately reducible to (5.3) upon substituting (5.6), suggests that substituting $\mathcal{H}$, then an alternative bilinear form will occur. By doing so, we find indeed an alternative bracket $\{F,G\}_H$ which describes the dynamics correctly via

$$\partial_t f = \{f,\mathcal{C}\}_H\,. \tag{5.9}$$

The explicit expression of this bilinear form is

$$\begin{aligned}\{F,G\}_H = \quad & -\int d^3x\Big\{\frac{1}{d_e^2}\mathbf{v}\cdot[(\nabla\times F_{\boldsymbol{\omega}})\times(\nabla\times G_{\boldsymbol{\omega}})] \\ & +[\mathbf{v}-d_i(\nabla\times\mathbf{B})]\cdot[(\nabla\times F_{\mathbf{B}^*})\times(\nabla\times G_{\mathbf{B}^*})] \\ & +(\nabla\times\mathbf{B})\cdot\big[(\nabla\times F_{\boldsymbol{\omega}})\times(\nabla\times G_{\mathbf{B}^*}) \\ & -(\nabla\times G_{\boldsymbol{\omega}})\times(\nabla\times F_{\mathbf{B}^*})\big]\Big\}\,. \end{aligned} \tag{5.10}$$

Immediately one can see that $\{F,\mathcal{H}\}_H = 0\ \forall F$, that is $\mathcal{H}$ and $\mathcal{C}$ have changed roles being now the Casimir and the Hamiltonian, respectively.

5.1.3 Translationally symmetric XMHD (tsXMHD)

Introducing a continuous spatial symmetry the description of the dynamics can be reduced to a 4-field model [53] since considering, e.g. translational symmetry, amounts to writing the 3D fields in the following 4-field representation

$$\mathbf{v} = v_z\hat{z} + \nabla\chi \times \hat{z}\,, \tag{5.11}$$

$$\boldsymbol{\omega} = \omega_z\hat{z} + \nabla v_z \times \hat{z}\,, \tag{5.12}$$

$$\mathbf{B}^* = B_z^*\hat{z} + \nabla\psi^* \times \hat{z}\,, \tag{5.13}$$

$$\mathbf{J} = J_z\hat{z} + \nabla B_z \times \hat{z}\,, \tag{5.14}$$

where $\omega_z = -\Delta\chi$ and $J_z = -\Delta\psi$. The functional derivatives in the reduced representation can be found as done in Chapter 2 (see [53] and [71])

$$F_{\mathbf{B}^*} = F_{B_z^*}\hat{z} - \nabla(\Delta^{-1}F_{\psi^*}) \times \hat{z}\,, \tag{5.15}$$

$$F_{\boldsymbol{\omega}} = F_{\omega_z}\hat{z} - \nabla(\Delta^{-1}F_{v_z}) \times \hat{z}\,, \tag{5.16}$$

and taking their curls yields

$$\nabla \times F_{\boldsymbol{\omega}} = F_{v_z}\hat{z} + \nabla F_{\omega_z} \times \hat{z}\,, \tag{5.17}$$

$$\nabla \times F_{\mathbf{B}^*} = F_{\psi^*} + \nabla F_{B_z^*} \times \hat{z}\,. \tag{5.18}$$

Substituting (5.17) and (5.18) into (5.7) one can readily find its translationally symmetric version to be

$$\{F,G,I\} = \int d^2x \left\{ \frac{1}{d_e^2} F_{v_z}[G_{\omega_z}, I_{\omega_z}] - d_i F_{\psi^*}[G_{B_z^*}, I_{B_z^*}] \right.$$
$$\left. + F_{\psi^*}[G_{\omega_z}, I_{B_z^*}] + I_{\psi^*}[F_{B_z^*}, G_{\omega_z}] + G_{v_z}[I_{B_z^*}, F_{B_z^*}] \right\} + \circlearrowright (F,G,I)\,, \tag{5.19}$$

where $[a,b] := (\nabla a \times \nabla b)\cdot\hat{z}$. With the help of the symmetric counterparts of $\mathcal{H}$ and $\mathcal{C}$ given by

$$\tilde{\mathcal{H}} = \frac{1}{2}\int d^2x \,\left(v_z^2 + \chi\omega_z + B_z B_z^* + \nabla\psi^* \cdot \nabla\psi\right)\,, \tag{5.20}$$

$$\tilde{\mathcal{C}} = \int d^2x \,\left(\psi^* B_z^* + d_e^2 v_z \omega_z\right)\,, \tag{5.21}$$

respectively, we retrieve the translationally symmetric XMHD dynamics via $\partial_t f = \{f, \tilde{\mathcal{H}}, \tilde{\mathcal{C}}\}$, that is

$$\partial_t v_z = [\chi, v_z] + [B_z, \psi^*]\,, \tag{5.22}$$

$$\partial_t \psi^* = [\chi, \psi^*] - d_i[B_z, \psi^*] + d_e^2[B_z, v_z]\,, \tag{5.23}$$

$$\partial_t \omega_z = [\chi, \omega_z] + [\psi, \Delta\psi] - d_e^2[B_z, \Delta B_z] \tag{5.24}$$

$$\partial_t B_z^* = [\chi, B_z^*] + [v_z, \psi] - d_i[\psi, \Delta\psi] + d_i d_e^2[B_z, \Delta B_z] + d_e^2[B_z, \omega_z]\,. \tag{5.25}$$

Similarly to the 3D case a reduced bilinear bracket can be constructed upon substituting $\tilde{\mathcal{H}}$ into (5.19) to find

$$\{F,G\}_{\tilde{\mathcal{H}}} = -\int d^2x \Big\{ d_e^{-2} v_z [F_{\omega_z}, G_{\omega_z}] + d_e^{-2}\chi\left([F_{v_z}, G_{\omega_z}] - [G_{v_z}, F_{\omega_z}]\right) \\ +(\chi - d_i B_z)\left([F_{\psi^*}, G_{B_z^*}] - [G_{\psi^*}, F_{B_z^*}]\right) \\ +(v_z - d_i J_z)[F_{B_z *}, G_{B_z^*}] + J_z\left([F_{B_z^*}, G_{\omega_z}] - [G_{B_z^*}, F_{\omega_z}]\right) \\ B_z\left([F_{\omega_z}, G_{\psi^*}] - [G_{\omega_z}, F_{\psi^*}] + [F_{B_z^*}, G_{v_z}] - [G_{B_z^*}, F_{v_z}]\right) \Big\}, \qquad (5.26)$$

with Eqs. (5.22)–(5.25) following from $\partial_t f = \{f, \tilde{\mathcal{C}}\}_{\tilde{\mathcal{H}}}$.

5.2 2D Magnetofluid models constructed via a priori imposition of conservation laws

As stated in the first, introductory chapter, the Casimirs introduce dynamical constraints and together with the Hamiltonian they determine the manifold on which the evolution of the dynamical system is restricted. In [127] and [128] the authors constructed Nambu-like brackets for the 2D ideal incompressible hydrodynamics and for the shallow water equations respectively, using differential 2-forms to impose orthogonality conditions arising from the conservation laws (CLs) of the respective models. To demonstrate the applicability of this idea to plasma fluid models we adopt an analogous approach in the context of the simple RMHD model that is essentially the 2D incompressible MHD. In addition we ascertain that the imposition of the conservation laws as orthogonality constraints, can be useful to derive models possessing Nambu-like and Poisson structures with the a priori definition of two ingredients: 1) the dynamical variables, 2) the functional quantities that are to be conserved by the dynamics. This also provides the freedom to select a subset of the original orthogonality conditions, so as to obtain models that do not conserve all of the ideal invariants, though they are nondissipative. Such an idea could potentially be linked with the concept of selective decay in magnetohydrodynamic turbulence, which assumes that the total energy is minimized, subject to the conservation of the helicities in the sense that one can incorporate additional contributions that break the ideal conservation of a particular invariant in order to regulate its decay individually, without affecting the remaining invariants as the diffusive terms do. Also, as it is pointed out in Subsection 5.2.5 some classes of those models can serve as useful regularizations of the original RMHD model in the sense that they prevent vorticity singularities having a nondissipative structure. Similar regularizations were introduced in [129] for the three-dimensional incompressible MHD equations.

We should clarify that we do not completely solve the inverse problem of conservation laws for models of the MHD form. The complete determination of the full set of models that respect given conservation laws would be a very tough task and so far we are not aware of any suitable methodology in order to address this problem. As we will see below, we shortcut by considering that the time derivatives of the dynamical variables assume a specific form compatible with the Lie-Poisson brackets of Hamiltonian systems such as RMHD [130], which is practically convenient for performing various manipulations.

5.2.1 Reduced MHD

Reduced MHD models are used to displace the usual 3D MHD equations when a strong guiding magnetic field $\mathbf{B}_0$ is present, because they are much more simpler in form and thus can be handled more conveniently. The Reduced MHD model can be rigorously derived by performing asymptotic expansion of the MHD equations with the ordering $L_\perp/L_\| \sim B_\perp/B_0 \sim v/v_A \sim \epsilon$ and ϵ being a small parameter [131]. Here $L_\perp$ and $L_\|$ are the characteristic length scales perpendicular and parallel to the guiding field respectively, $B_\perp$ and B_0 are the corresponding magnetic field magnitudes, v is the magnitude of the velocity and v_A is the Alfvén velocity. Alternatively one can just confine the dynamics to take place on the plane perpendicular to the guiding field $\mathbf{B}_0 = B_0\hat{z}$, i.e. to express $\mathbf{B}$ and $\mathbf{v}$ as

$$\mathbf{B} = \nabla\psi \times \hat{z}\,, \qquad \mathbf{v} = \nabla\chi \times \hat{z}\,, \tag{5.27}$$

where ψ is the magnetic flux function and χ is the velocity stream function. Assuming that the plasma is incompressible, in the sense that the mass density is uniform throughout the plasma volume, one can derive from the general momentum and induction equations of the MHD model, the following dynamical equations, defined on a bounded domain $D \subset \mathbb{R}^2$,

$$\partial_t\omega = [\chi, \omega] + [J, \psi]\,, \tag{5.28}$$

$$\partial_t\psi = [\chi, \psi]\,, \tag{5.29}$$

where $J := -\Delta\psi$ and $\omega := -\Delta\chi$, are the magnitudes of the current density and vorticity respectively and $[a, b] := (\partial_x a)\,(\partial_y b) - (\partial_x b)\,(\partial_y a)$ is the Jacobi-Poisson bracket. Here the subscripts z are omitted since $\mathbf{J}$ and $\boldsymbol{\omega}$ have only $z-$components.

In [130] the authors proved that the RMHD model, and also its compressible counterpart, possess a noncanonical Hamiltonian structure, since the dynamics can be expressed in terms of a degenerate Poisson bracket and a Hamiltonian, as follows

$$\partial_t\omega = \{\omega, \mathcal{H}\}\,, \qquad \partial_t\psi = \{\psi, \mathcal{H}\}\,, \tag{5.30}$$

where the Hamiltonian $\mathcal{H}$ is

$$\mathcal{H}[\omega,\psi]=\frac{1}{2}\int_D d^2x\,\left(|\nabla\chi|^2+|\nabla\psi|^2\right)=-\frac{1}{2}\int_D d^2x\,(\omega\Delta^{-1}\omega+\psi\Delta\psi)\,, \tag{5.31}$$

and the Poisson bracket is given by

$$\{F,G\}=\int_D d^2x\,\{\omega[F_\omega,G_\omega]+\psi\left([F_\psi,G_\omega]-[G_\psi,F_\omega]\right)\}\,. \tag{5.32}$$

This bracket has two families of Poisson-commuted functionals, i.e. Casimir invariants, given by

$$\mathcal{C}=\int_D d^2x\,\omega\mathcal{F}(\psi)\,,\quad \mathcal{M}=\int_D d^2x\,\mathcal{G}(\psi)\,, \tag{5.33}$$

where $\mathcal{F}$ and $\mathcal{G}$ are arbitrary functions. The Casimir $\mathcal{C}$ is a cross-helicity-like functional while $\mathcal{M}$ expresses the conservation of magnetic flux. Therefore the incompressible RMHD model has three general CLs, expressed through the preservation of the functionals $\mathcal{H}$, $\mathcal{C}$ and $\mathcal{M}$. Here, the structure of the dynamical equations (5.28)-(5.29) and of the Poisson bracket (5.32) indicate the conservation laws. In what follows we try to reverse this procedure, i.e. with the CLs at hand, we construct the dynamical equations that conserve the associated invariants.

5.2.2 Dynamics via orthogonality conditions

Let us assume that we have a continuous system bounded in a 2D domain D and described by dynamical variables $X=(X_1,...,X_N)$. Also, assume that the system exhibits conservation of a set of M quantities $Y_1[X],...,Y_M[X]$ expressed as functionals defined on phase space. The conservation of $Y_i[X]$, $i=1,...,M$ implies

$$\begin{aligned} c_1:\ \frac{dY_1[X]}{dt} &= \int_D d^2x\,\frac{\delta Y_1}{\delta X_i}\partial_t X_i=0\,,\\ &\ \vdots \\ c_M:\ \frac{dY_M[X]}{dt} &= \int_D d^2x\,\frac{\delta Y_M}{\delta X_i}\partial_t X_i=0\,. \end{aligned} \tag{5.34}$$

Equations (5.34) define a set of M orthogonality conditions c_i, $i=1,...,M$, of the vectors

$$\mu_i=\left(\frac{\delta Y_i}{\delta X_1},...,\frac{\delta Y_i}{\delta X_N}\right)\,,\quad i=1,...,M\,, \tag{5.35}$$

with the vector

$$\sigma=(\partial_t X_1,...,\partial_t X_N)^T\,. \tag{5.36}$$

Since μ_i are known, then in principle the orthogonality conditions (5.34) can be exploited in order to find the components of σ. In the case of 2D hydrodynamics the orthogonality conditions correspond to the conservation of kinetic energy $\mathcal{K}$ and enstrophy $\mathcal{E}$. In [127] the authors imposed conveniently those conditions using differential 2-forms identifying $\mathcal{K}_\omega$ as a 0-form μ_1 and $\mathcal{E}_\omega$ as a 0-form μ_2. Then if $\partial_t\omega = d\mu_1 \wedge d\mu_2$ is assumed to be exact, then it is automatically orthogonal to μ_1 and μ_2.

Here, a similar approach is adopted, i.e. we start by considering the conservation laws as orthogonality conditions like those in (5.34) that act as constraints on the dynamics, in order to construct a 2D continuum model, with dynamical variables the vorticity and the poloidal magnetic flux function, that conserves $\mathcal{H}$, $\mathcal{C}$ and $\mathcal{M}$ as given by (5.33). To do so let us define the following vectors

$$
\begin{aligned}
\xi &:= (\omega,\, \psi)\ ,\\
f &:= (\mathcal{H}_\omega,\, \mathcal{H}_\psi) = (\chi,\, J)\ ,\\
g &:= (\mathcal{C}_\omega,\, \mathcal{C}_\psi) = \big(\mathcal{F}(\psi),\, \omega\mathcal{F}'(\psi)\big)\ ,\\
h &:= (\mathcal{M}_\omega,\, \mathcal{M}_\psi) = \big(0,\, \mathcal{G}'(\psi)\big)\ ,\\
\sigma &:= \partial_t\xi^T = (\partial_t\omega,\, \partial_t\psi)^T\ ,
\end{aligned}
\tag{5.37}
$$

The time invariance of $\mathcal{H}$, $\mathcal{C}$ and $\mathcal{M}$ yields

$$
\begin{aligned}
c_1 &: \frac{d\mathcal{H}}{dt} = \int_D d^2x\, \left[\mathcal{H}_\omega(\partial_t\omega) + \mathcal{H}_\psi(\partial_t\psi)\right] = 0\,,\\
c_2 &: \frac{d\mathcal{C}}{dt} = \int_D d^2x\, \left[\mathcal{C}_\omega(\partial_t\omega) + \mathcal{C}_\psi(\partial_t\psi)\right] = 0\,,\\
c_3 &: \frac{d\mathcal{M}}{dt} = \int_D d^2x\, \left[\mathcal{M}_\omega(\partial_t\omega) + \mathcal{M}_\psi(\partial_t\psi)\right] = 0\,,
\end{aligned}
\tag{5.38}
$$

Those orthogonality conditions can be expressed with the use of Eqs. (5.37) as

$$
\begin{aligned}
c_1 &: \int_D d^2x\, f_i\sigma_i = 0\,,\\
c_2 &: \int_D d^2x\, g_i\sigma_i = 0\,,\\
c_3 &: \int_D d^2x\, h_i\sigma_i = 0\,, \qquad i = 1,2\,.
\end{aligned}
\tag{5.39}
$$

In noncanonical Hamiltonian theories involving 2D Lie-Poisson brackets the dynamics is governed by Hamilton's equations of the form

$$
\partial_t F = \{F, \mathcal{H}\}\,,
\tag{5.40}
$$

with $\{F,G\}$ being a Lie-Poisson bracket with general form

$$\{F,G\} = \int_D d^2x\, \xi^k [F_{\xi^m}, G_{\xi^n}] W_k^{m,n}\,, \tag{5.41}$$

where ξ is the set of the dynamical variables and $W_k^{m,n}$ are constants. Note that repeated indices imply summation. Equations (5.40) and (5.41) indicate that the time independent parts of the evolution equations can be written as linear combinations of Jacobi-Poisson brackets between the various dynamical variables and the functional derivatives of the Hamiltonian. Therefore we assume that the building block of $\sigma_{1,2}$ is the Jacobian bracket i.e.

$$\sigma_i = \gamma_{ijk}\,[f_j, \xi_k]\,, \quad i,j,k = 1,2\,. \tag{5.42}$$

Note that this ansatz for σ_i possibly excludes models that do respect the given CLs. However, it is a reasonable choice since it is consistent with the Lie-Poisson Hamiltonian framework and is convenient in order to carry out manipulations that easily result in systems of equations that preserve the Casimirs. In fact exploiting the identity

$$\int_D d^2x\, a[b,c] = \int_D d^2x\, c[a,b] = \int_D d^2x\, b[c,a]\,, \tag{5.43}$$

which holds for appropriate boundary condition, e.g., periodic, we can find that constraints c_1, c_2 and c_3 induce the following sets of conditions for the parameters γ_{ijk} with $i,j,k = 1,2$, which have to hold true for the dynamical variables ψ and ω to be independent,

$$\begin{aligned}
c_1 &: \{\gamma_{121} = \gamma_{211}\,,\ \gamma_{122} = \gamma_{212}\}\,,\\
c_2 &: \{\gamma_{111} = \gamma_{212}\,,\ \gamma_{121} = \gamma_{222}\} \text{ for } \mathcal{F}''(\psi) = 0\,,\\
&\quad \{\gamma_{111} = \gamma_{212}\,,\ \gamma_{121} = \gamma_{222}\,,\ \gamma_{211} = 0\,,\ \gamma_{221} = 0\} \text{ for } \mathcal{F}''(\psi) \neq 0\,,\\
c_3 &: \{\gamma_{211} = 0\,,\ \gamma_{221} = 0\}\,.
\end{aligned} \tag{5.44}$$

The remaining parameters γ_{ijk} are arbitrary. Constraint c_3, stemming from magnetic flux conservation, contributes in determining the parametric conditions only when $\mathcal{F}(\psi)$ is linear in ψ. In all other cases, imposing the conservation of the cross helicity functional ensures the conservation of the magnetic flux as well. Therefore, the trajectory of RMHD dynamics is determined only by the intersection of the energy and the cross helicity level sets in phase space if $\mathcal{F}''(\psi) \neq 0$.

The above results imply that the conservation of the RMHD Casimirs require

$$\begin{aligned}
\gamma_{111} = \gamma_{212} = \gamma_{122} \equiv \epsilon_1\,, \quad \gamma_{112} \equiv \epsilon_2\,,\\
\gamma_{121} = \gamma_{211} = \gamma_{221} = \gamma_{222} = 0,
\end{aligned} \tag{5.45}$$

where ϵ_1 and ϵ_2 are arbitrary parameters, introduced to simplify notation. In view of (5.45) and (5.42) we take

$$\begin{aligned} \partial_t \omega &= ([\mathcal{H}_\psi, \psi] + [\mathcal{H}_\omega, \omega]) + \epsilon[\mathcal{H}_\omega, \psi]\,, &(5.46)\\ \partial_t \psi &= [\mathcal{H}_\omega, \psi]\,. &(5.47) \end{aligned}$$

where parameter ϵ_1 was absorbed by rescaling the time variable, and ϵ_2 was renamed. Evaluating the functional derivatives we obtain

$$\begin{aligned} \partial_t \omega &= ([J, \psi] + [\chi, \omega]) + \epsilon[\chi, \psi]\,, &(5.48)\\ \partial_t \psi &= [\chi, \psi]\,. &(5.49) \end{aligned}$$

It is easy to corroborate that model (5.48)-(5.49) conserves the energy $\mathcal{H}$ and the Casimirs $\mathcal{C}$ and $\mathcal{M}$ as given in (5.33). This generalized model includes RMHD as a special case since the latter is recovered for $\epsilon = 0$. Note that conditions (5.45) can be interpreted as follows: the inclusion of $[J, \omega]$ in any of the dynamical equations of the model violates the conservation laws. In addition, the evolution of ψ is coerced to not involve dependence on J and ω.

5.2.3 Poisson and Nambu bracket description

To construct the Poisson and Nambu brackets for (5.48)–(5.49) we have just to consider the time evolution of an arbitrary functional $F = F[\omega, \psi]$

$$\partial_t F = \int_D d^2x \; [F_\omega(\partial_t \omega) + F_\psi(\partial_t \psi)] \;, \tag{5.50}$$

and use equations (5.46), (5.47), with the arbitrary functional G replacing the Hamiltonian, to obtain

$$\{F, G\} = \{F, G\}_{RMHD} + \epsilon \int_D d^2x\, \psi[F_\omega, G_\omega]\,, \tag{5.51}$$

where $\{F, G\}_{RMHD}$ is given by (5.32). Bracket (5.51) satisfies the Jacobi identity since the matrices W^n, $n = 1, 2$ in (5.41) pairwise commute [132]. For deriving the Nambu formalism of system (5.48)–(5.49) we need just to observe that $\psi = \bar{\mathcal{C}}_\omega$ and $\omega = \bar{\mathcal{C}}_\psi$ where $\bar{\mathcal{C}} = \mathcal{C}$ for $\mathcal{F}(\psi) = \psi$. Making this substitution we convert the Lie-Poisson bracket (5.51) to the following trilinear bracket

$$\begin{aligned} \{F, G, Z\} := \int_D d^2x\, \{ & F_\omega[G_\psi, Z_\omega] + F_\omega[G_\omega, Z_\psi] \\ & + F_\psi[G_\omega, Z_\omega] + \epsilon F_\omega[G_\omega, Z_\omega] \}\,. \end{aligned} \tag{5.52}$$

The dynamics can be described by $\partial_t F = \{F, \mathcal{H}, \bar{\mathcal{C}}\}$. The bracket is completely antisymmetric in its three arguments in view of identity (5.43) and the antisymmetry of the Jacobi-Poisson bracket $[f, g] = -[g, f]$. As mentioned earlier, this property is helpful in constructing numerical schemes that preserve to high precision the energy and the cross helicity [125].

5.2.4 Canonical description

One may derive a canonical description of system (5.48)–(5.49) by expressing the vorticity ω and the flux function ψ in terms of Clebsch potentials. Canonical descriptions of the RMHD model were derived in [130] and in [133]. The former derivation needs four Clebsch potentials while the latter only two and both assumed the vorticity to be a Clebsch 2-form $\omega = [P, Q]$. In [133] the authors found two suitable parametrization schemes for ψ, namely $\psi = P^\alpha Q^\beta$ and $\psi = P^\alpha + Q^\beta$. Following [133] we use $\psi = P^\alpha Q^\beta$ for our generic model and also we make a necessary modification in parameterizing ω

$$\begin{aligned} \omega &= [P, Q] + \epsilon P^\alpha Q^\beta \,, \\ \psi &= P^\alpha Q^\beta \,. \end{aligned} \tag{5.53}$$

The Hamiltonian takes the form

$$\begin{aligned} \mathcal{H} = -\frac{1}{2} \int d^2x \{ & [P, Q] \Delta^{-1} [P, Q] \\ & + \epsilon P^\alpha Q^\beta \Delta^{-1} [P, Q] + \epsilon [P, Q] \Delta^{-1} (P^\alpha Q^\beta) \\ & + \epsilon^2 P^\alpha Q^\beta \Delta^{-1} (P^\alpha Q^\beta) + P^\alpha Q^\beta \Delta (P^\alpha Q^\beta) \} \,, \end{aligned} \tag{5.54}$$

and the canonical Hamilton's equations are

$$\partial_t \begin{pmatrix} P \\ Q \end{pmatrix} = \mathcal{J}_c \begin{pmatrix} \mathcal{H}_P \\ \mathcal{H}_Q \end{pmatrix} \,, \tag{5.55}$$

where $\mathcal{J}_c$ represents the so-called cosymplectic operator (see Chapter 1)

$$\mathcal{J}_c = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \,. \tag{5.56}$$

In view of (5.54), Hamilton's equations (5.55) take the form

$$\begin{aligned} \partial_t P &= [\chi, P] + \beta P^\alpha Q^{\beta-1} J + \epsilon \beta P^\alpha Q^{\beta-1} \chi \,, \\ \partial_t Q &= [\chi, Q] - \alpha P^{\alpha-1} Q^\beta J - \epsilon \alpha P^{\alpha-1} Q^\beta \chi \,, \end{aligned} \tag{5.57}$$

where $J = -\Delta(P^\alpha Q^\beta)$ and $\chi = -\Delta^{-1}[P, Q] - \epsilon \Delta^{-1}(P^\alpha Q^\beta)$. Using Eqs. (5.53) and (5.57) and exploiting the Jacobi identity, the original system (5.48)–(5.49) can be recovered. The cross-helicity for $\mathcal{F}(\psi) = \psi$ is given by

$$\mathcal{C} = \int_D d^2x \left(\epsilon P^{2\alpha} Q^{2\beta} + P^{\alpha} Q^{\beta} [P, Q] \right) . \tag{5.58}$$

The first term is a conserved quantity due to the conservation of $\mathcal{M}$, therefore the second term, which is the RMHD cross helicity, is also conserved. A reason for writing the system (5.48)-(5.49) in terms of Clebsch potentials is to see how the addition of the ϵ-term alternates the form of the Hamiltonian. The Clebsch-parameterized Hamiltonian of our generic model is different from its RMHD counterpart (obtained by setting $\epsilon = 0$) albeit when expressed in noncanonical Eulerian variables they are identical. This difference is a consequence of the fact that in canonical description any complexity is removed from the Poisson bracket and is transfered into the Hamiltonian. Note that although the Hamiltonian acquires an explicit dependence on the parameter ϵ, the Casimirs do not contain this parameter. Another reason is that the canonicalization of a noncanonical Hamiltonian system may be useful for numerical studies as well, because the symplectic and conservative algorithms for canonical Hamiltonian mechanics are much more developed than those in the noncanonical framework.

5.2.5 Families of reduced models respecting two out of the three original CLs

$\mathcal{H}, \bar{\mathcal{C}}$ conserving models

From conditions (5.44) one deduces that for $\mathcal{F}''(\psi) \neq 0$ the most general model that conserves $\mathcal{H}$ and $\mathcal{C}$ is model (5.48)–(5.49). However for linear $\mathcal{F}(\psi)$ there are a lot of new possibilities since there are two additional arbitrary parameters. Conditions (5.44) imply that for a family of 2D hydromagnetic models, with dynamical variables the vorticity ω and the magnetic flux ψ, that conserve only the Energy $\mathcal{H}$ and the linear cross-helicity $\bar{\mathcal{C}}$, the coefficients in the expansions (5.42) should be

$$\begin{aligned} \gamma_{111} = \gamma_{122} = \gamma_{212} \equiv \epsilon_1 \,, \quad \gamma_{112} \equiv \epsilon_2 \,, \\ \gamma_{121} = \gamma_{211} = \gamma_{222} \equiv \epsilon_3 \,, \quad \gamma_{221} \equiv \epsilon_4 \,. \end{aligned} \tag{5.59}$$

Conditions (5.59) with (5.42) lead to the following expansions

$$\begin{aligned} \partial_t \omega &= \epsilon_1 \left([\mathcal{H}_\omega, \omega] + [\mathcal{H}_\psi, \psi] \right) + \epsilon_2 [\mathcal{H}_\omega, \psi] + \epsilon_3 [\mathcal{H}_\psi, \omega] \,, \\ \partial_t \psi &= \epsilon_1 [\mathcal{H}_\omega, \psi] + \epsilon_4 [\mathcal{H}_\psi, \omega] + \epsilon_3 \left([\mathcal{H}_\omega, \omega] + [\mathcal{H}_\psi, \psi] \right) \,, \end{aligned} \tag{5.60}$$

which result in the generalized model

$$\partial_t \omega = \epsilon_1 \left([\chi, \omega] + [J, \psi] \right) + \epsilon_2 [\chi, \psi] + \epsilon_3 [J, \omega] \,,$$

$$\partial_t \psi = \epsilon_1[\chi, \psi] + \epsilon_3 \left([\chi, \omega] + [J, \psi]\right) + \epsilon_4[J, \omega] \,. \tag{5.61}$$

Ordinary RMHD is recovered for $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) = (1, 0, 0, 0)$. Setting $\epsilon_1 = 1$ (or rescaling the time variable so as to absorb ϵ_1) in order to retain the RMHD core, we can build, apart from model (5.61), six extensions of RMHD that conserve the energy and the linear cross-helicity. In the generic case represented by (5.61), $\mathcal{M}$ evolves as

$$\frac{d\mathcal{M}}{dt} = \int_D d^2x \, (\epsilon_3\chi + \epsilon_4 J)\mathcal{G}''(\psi)[\omega, \psi] \,, \tag{5.62}$$

while evolution of the rest members of the family of cross helicity invariants (5.33) is given by

$$\frac{d\mathcal{C}}{dt} = \int_D d^2x \, (\epsilon_3\chi + \epsilon_4 J)\omega\mathcal{F}''(\psi)[\omega, \psi] \,. \tag{5.63}$$

Equation (5.63) indicates that the conservation of $\mathcal{C}$ is possible either if $\mathcal{F}''(\psi) = 0$, which is the case we discuss in this subsection, or if $\epsilon_3 = \epsilon_4 = 0$, which results in system (5.48)–(5.49) of the previous section.

The generalized model (5.61) can be cast into a Hamiltonian form in terms of the Hamiltonian (5.31) and a Lie-Poisson bracket with Hamilton's equations stemming from the substitution of (5.60) into (5.50). By this procedure we find that the Lie-Poisson bracket is

$$\begin{aligned} \{F, G\} = \epsilon_1\{F, G\}_{RMHD} + \int_D d^2x \Big\{ & \epsilon_2\psi[F_\omega, G_\omega] + \epsilon_4\omega[F_\psi, G_\psi] \\ & + \epsilon_3\omega\left([F_\omega, G_\psi] - [G_\omega, F_\psi]\right) + \epsilon_3\psi[F_\psi, G_\psi] \Big\} \,. \end{aligned} \tag{5.64}$$

However, the Jacobi identity is satisfied only if $\epsilon_1\epsilon_3 = \epsilon_2\epsilon_4$. The Nambu description can be obtained similarly with subsection 5.2.3, resulting into a completely antisymmetric three-bracket. Note that the requirement $\epsilon_1\epsilon_3 = \epsilon_2\epsilon_4$ implies that there are only three nontrivial Hamiltonian extensions of RMHD that conserve $\mathcal{H}$ and $\mathcal{C}$. As an example let us consider the model $(1, 0, 0, \epsilon_4)$

$$\begin{aligned} \partial_t\omega &= [\chi, \omega] + [J, \psi] \,, \\ \partial_t\psi &= [\chi, \psi] + \epsilon_4[J, \omega] \,. \end{aligned} \tag{5.65}$$

One can easily identify that (5.65), in addition to $\mathcal{H}$ and $\mathcal{C}$, conserves also a generalized enstrophy

$$\tilde{\mathcal{E}} = \int_D d^2x \, \left(\psi^2 + \epsilon_4\omega^2\right) \,, \tag{5.66}$$

which is a Casimir of the Poisson bracket (5.64) with $\epsilon_2 = \epsilon_3 = 0$. The conservation of this "enstrophy" functional implies that, if $\epsilon_4 > 0$, this model converts enstrophy to magnetic flux and vice versa, which means that both $\int_D d^2x\,\omega^2$ and $\int_D d^2x\,\psi^2$ are bounded during the evolution, since the maximum value they can attain is the initial value of $\mathcal{E}$. Therefore the addition of the term $\epsilon_4[J,\omega]$ with positive ϵ_4 in the induction equation, regularize the RMHD system at least in preventing possible unbounded behavior of the vorticity. Usually such unbounded behavior is remedied by the inclusion of dissipative terms. However dissipation destroys time reversibility and the various conservation laws.

Before proceeding to the next category of models let us make an additional remark: the Poisson bracket (5.64) with $\epsilon_2 = \epsilon_3 = 0$ and $\epsilon_1 = 1$, incidentally has the same form with the Poisson bracket of a generalized model with finite electron inertia and ion sound Larmor radius effects in 2D geometry [134]. Although the bracket has the same form, the evolution equations and the Hamiltonian in the above referenced model are different from (5.65) and (5.31) respectively. One can see though that system (5.65) can be converted to the model with electron inertial and ion sound Larmor radius effects by performing the following transformation $(\psi \to \psi^*, \chi \to \chi^*, \omega \to \omega, J \to J)$, where $\psi^* = \psi + d_e^2 J$, $\chi^* = \chi + \rho_s^2\omega$, and identifying $\epsilon_4 = \rho_s^2 d_e^2$. Here d_e is the electron skin depth and ρ_s the ion sound Larmor radius. This transformation changes the stream functions but not the corresponding "vorticities", which means that the fourth and higher order spatial derivatives (associated with very small length scales) are neglected. Hence, bracket (5.64) remains identical in form when written in terms of ω and ψ^* but the Hamiltonian changes. It is also noted that a similar bracket has been derived for describing the perpendicular dynamics in a 4-field gyrofluid model in [135].

$\mathcal{H}, \mathcal{M}-$ conserving models

To construct models that conserve $\mathcal{H}$ and $\mathcal{M}$ we need to employ conditions c_1 and c_3. According to (5.44) the imposition of the aforementioned orthogonality conditions leads to

$$\begin{aligned} \gamma_{111} \equiv \epsilon_1\,, \quad \gamma_{112} \equiv \epsilon_2\,, \quad \gamma_{122} = \gamma_{212} = \epsilon_3\,, \\ \gamma_{222} \equiv \epsilon_4\,, \quad \gamma_{211} = \gamma_{221} = \gamma_{121} = 0\,, \end{aligned} \tag{5.67}$$

that is, the general model that conserves $\mathcal{H}$ and $\mathcal{M}$ is

$$\begin{aligned} \partial_t\omega &= \epsilon_1[\chi,\omega] + \epsilon_2[\chi,\psi] + \epsilon_3[J,\psi]\,, \\ \partial_t\psi &= \epsilon_3[\chi,\psi] + \epsilon_4[J,\psi]\,. \end{aligned} \tag{5.68}$$

RMHD is recovered for $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) = (1, 0, 1, 0)$. For the general form of equations (5.68), the evolution of cross-helicity is given by

$$\frac{d\mathcal{C}}{dt} = (\epsilon_1 - \epsilon_3) \int_D d^2x\, \mathcal{F}(\psi)[\chi, \omega] + \epsilon_4 \int_D d^2x\, \mathcal{F}(\psi)[\omega, J]\,. \tag{5.69}$$

As for the Hamiltonian description, employing the usual procedure of the previous subsections for model (5.68) we identify that the dynamics is described by (5.30) with the following Poisson bracket

$$\begin{aligned} \{F, G\} = \int_D d^2x\, \{&(\epsilon_2\psi + \epsilon_1\omega)[F_\omega, G_\omega] \\ &+\epsilon_3\psi\left([F_\psi, G_\omega] - [G_\psi, F_\omega]\right) + \epsilon_4\psi[F_\psi, G_\psi]\}\,. \end{aligned} \tag{5.70}$$

Bracket (5.70) is clearly antisymmetric but it satisfies the Jacobi identity only if $\epsilon_3^2 - \epsilon_1\epsilon_3 - \epsilon_2\epsilon_4 = 0$ with roots $\epsilon_3^{\pm} = \left(\epsilon_1 \pm \sqrt{\epsilon_1^2 + 4\epsilon_2\epsilon_4}\right)/2$. Under this condition, model (5.68) has a Hamiltonian structure, with Hamiltonian functional given by (5.31) and a Poisson bracket given by (5.70), which possess apart from $\mathcal{M}$, an additional Casimir having the form of a generalized cross helicity

$$\tilde{\mathcal{C}} = \int_D d^2x\, \omega\left(\psi + \frac{\mu_\pm}{2}\omega\right)\,, \tag{5.71}$$

where $\mu_\pm = \left[\left(\epsilon_1 \pm \sqrt{\epsilon_1^2 + 4\epsilon_2\epsilon_4}\right)/(2\epsilon_4)\right]^{-1}$. It is known that the absolute value of cross helicity has the total energy $\mathcal{H}$ as an upper bound[1] therefore if $\mu_\pm > 0$ then the enstrophy is prevented from exhibiting unbounded growth.

A trilinear bracket formulation is also possible upon recognizing that $\omega = \tilde{\mathcal{C}}_\psi$ and $\psi = \tilde{\mathcal{C}}_\omega - \mu_\pm\tilde{\mathcal{C}}_\psi$. Substituting ψ and ω in (5.70) by these relations, we can find a completely antisymmetric trilinear bracket as in Subsection 5.2.3. The dynamics is described by means of this bracket along with the Hamiltonian and the Casimir $\tilde{\mathcal{C}}$.

$\mathcal{C}, \mathcal{M}-$ conserving models

To abandon the requirement for the energy to be conserved, we impose the constraints c_2 and c_3 only. From conditions (5.44) we take

$$\begin{aligned} &\gamma_{111} = \gamma_{212} \equiv \epsilon_1\,, \quad \gamma_{112} \equiv \epsilon_2\,, \quad \gamma_{122} \equiv \epsilon_3\,, \\ &\gamma_{121} = \gamma_{222} \equiv \epsilon_4\,, \quad \gamma_{211} = \gamma_{221} = 0\,, \end{aligned} \tag{5.72}$$

that is we obtain the following generalized model

$$\partial_t\omega \;=\; \epsilon_1[\chi, \omega] + \epsilon_2[\chi, \psi] + \epsilon_3[J, \psi] + \epsilon_4[J, \omega]\,,$$

[1] This can be proved by combining the Schwartz and the Poincaré inequalities to derive an inequality that relates the modulus of cross helicity with the sum of the magnetic and kinetic energies

$$\partial_t \psi \;\; = \;\; \epsilon_1[\chi, \psi] + \epsilon_4[J, \psi] \,. \tag{5.73}$$

RMHD corresponds to $(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) = (1, 0, 1, 0)$. For the generic model (5.73) the energy evolution is given by

$$\frac{d\mathcal{H}}{dt} = (\epsilon_3 - \epsilon_1) \int_D d^2x \, \chi[J, \psi] + \epsilon_4 \int_D d^2x \, \chi[J, \omega] \,. \tag{5.74}$$

System (5.73) conserves the entire families of $\mathcal{C}$ and $\mathcal{M}$ as given by (5.33). As an example of a 2D hydromagnetic model that exhibits a selective preservation of the two Casimirs, consider the RMHD generalization $(1, 0, 1, \epsilon_4)$, i.e.

$$\begin{aligned} \partial_t \omega \;\; &= \;\; [(\chi + \epsilon_4 J), \omega] + [J, \psi] \,, \\ \partial_t \psi \;\; &= \;\; [(\chi + \epsilon_4 J), \psi] \,. \end{aligned} \tag{5.75}$$

The only difference of (5.75) compared to (5.28)–(5.29) is that the former involves in the advection of ω and ψ the current density J. This means that small scale structures intervene in the advection of ω and ψ, indicating that such or similar models may have some practical implementations in parameterizing short length scale physics, which cannot be described adequately by the RMHD equations.

Chapter 6

Summary, conclusions and future prospects

6.1 Summary and conclusions

The main body of this thesis consists of an attempt to fulfill the four steps described in Subsection 1.3.3 for the extended MHD model, in the presence of a continuous spatial symmetry. These are i) the derivation of a noncanonical Poisson bracket $\{F, G\}$, governing the dynamics of the system with a Hamiltonian functional, ii) the computation of the corresponding Casimir invariants from $\{F, \mathcal{C}\} = 0$, iii) the derivation of equilibrium equations from the energy-Casimir variational principle and iv) the derivation of sufficient stability conditions. The first and second are fulfilled in Chapter 2, where the three-dimensional Poisson bracket is reduced to a helically symmetric one upon employing the chain rule for functional derivatives. The new bracket governs the helically symmetric XMHD dynamics, which for incompressible plasmas can be reduced to a 4-field model. The Casimirs are computed by a systematic procedure, which is clearly presented. Also, their HMHD, IMHD and MHD limits are recovered. Then in Chapter 3 the XMHD Casimirs and the Hamiltonian were used to derive, via the energy-Casimir variational principle, the equilibrium equations of helically symmetric XMHD. This symmetry makes both the dynamical and equilibrium equations more involved than the corresponding translationally symmetric equations, due to the presence of a scale factor $k = (\ell^2 + n^2 r^2)^{-1/2}$ and new terms stemming from helical symmetry. The equilibrium equations were manipulated further for two simpler cases: first was the axisymmetric, barotropic and incompressible XMHD and HMHD and second the helically symmetric barotropic and incompressible HMHD. Both systems with barotropic closures were cast in Grad-Shafranov-Bernoulli (GSB) forms, which describe completely the respective equilibria.

The axisymmetric HMHD GSB equations are integrated numerically using a finite difference solver which employs the iterative SOR method to solve the system resulting from the discretization of the partial differential equation for ψ, while the differential equation for φ and the Bernoulli equation are treated as algebraic equations. The solution was pursued on a D-Shaped domain, enclosed by a diverted boundary with lower x-point and ITER-relevant geometric characteristics. A pressure-related free function was chosen appropriately so as a pressure pedestal to be formed. The results show accumulation of sheared poloidal flow and toroidal current density in the external transport barrier region due to the steep density gradients therein. It is also observed that despite the fact that the influence of the Hall contributions on several equilibrium quantities of interest (pressure, mass density, current density, etc) is rather weak, it strongly affects the flow profiles and results in separation of magnetic and ion flow surfaces. Therefore, we conclude that when computing equilibria with strong flows that are to be used as initial conditions, for example in transport and turbulence studies, one should probably employ two fluid models such as those used in the present thesis. Even if the Hall parameter is small, strong flows may induce separation of the ionic surfaces as predicted by the Hall MHD model.

In the incompressible case, Bernoulli's equation can no longer be derived via the standard EC principle but one has to return to the primary equations of the model. The Bernoulli equation decouples from the equilibrium PDE system, becoming a secondary condition for the computation of the pressure. As an example, a particular case of equilibria was studied by means of an analytical solution. The application concerns an incompressible, helically symmetric plasma described by HMHD, for which we derived an analytic double-Beltrami solution and constructed an equilibrium configuration with nonplanar helical axis that can be regarded as a straight-stellarator-like equilibrium.

The XMHD equilibrium equations are new in literature and therefore, their properties are not yet elucidated. One feature of particular importance is the classification of the equilibrium PDEs. We examined this problem by deriving the ellipticity condition for the complete system and by further investigating some special cases. It turned out that the quasineutrality assumption together with the inclusion of electron inertia are of importance for the final form of the ellipticity condition. We derived a necessary and sufficient ellipticity condition from which a simplified sufficient condition can be deduced. The latter becomes necessary under certain assumptions, indicating that electron inertia lowers the threshold of the maximum poloidal center of mass velocity for the system to be elliptic. In particular, the electron inertial contribution may become considerable within regions of low mass density. Also, we found that in the context of XMHD, in principle, even static equilibrium equations can become hyperbolic, a consequence of the finite electron inertia.

In Chapter 4 we derived sufficient stability criteria, exploiting the Hamiltonian structure of the XMHD model via the EC and DA methods. In addition, Lagrangian stability is studied within a mixed Lagrangian-Eulerian description. Using the EC method we ascertained that indefinite terms appear in the second variation of the EC functional, occurring due to the vorticity-magnetic field coupling induced by the form of the Casimir invariants. We side-stepped this problem by either considering axisymmetric equilibria with purely toroidal flow or special perturbations, assumptions that enable the removal of the indefiniteness. For the special case of axisymmetric HMHD equilibria with purely toroidal flow, we implemented the corresponding stability criterion for ITER-like Tokamak equilibria. It turned out that for our numerical equilibria, the "compressible" part of the criterion is satisfied only within a narrow region in the high field side of the configuration if $\beta > 1\%$. Also, it is observed that increasing the Hall parameter is beneficial for stability, improving the stability diagrams.

To study stability under three dimensional perturbations we employed the DA method, which furthermore can be applied on the study of generic equilibria by restricting the perturbations to be tangent on the Casimir leaves. We found that the resulting criterion has a smooth MHD limit in contrast with the EC criteria. Also, its HMHD and MHD limits are consistent with previous studies.

We applied also Lagrangian stability analysis for the quasineutral, two-fluid model written in XMHD-like variables, namely the Lagrangian counterparts of the center of mass velocity and current density. Subsequently employing the Lagrange-Euler map and upon performing a Legendre transformation we found the governing Hamiltonian for linear dynamics in the Eulerian viewpoint. Considering massless electrons, the definition of one of the two canonical momenta led to a relation between the perturbed magnetic potential and canonical variables. Requiring this relation to be preserved by the dynamics, gave rise to a dynamical constraint; whence we found the solution to the perturbed induction equation, namely $\mathbf{B}_1 = \nabla \times [(\boldsymbol{\zeta} - d_i \boldsymbol{\eta}) \times \mathbf{B}]$. To our knowledge, this result has never been obtained by such methods before. In addition, by this procedure we generalized the HMHD energy principle of [15] so as to include the electron entropy and pressure contributions.

In Chapter 5, alternative bracket formulations for the incompressible XMHD equations were constructed, either using trilinear brackets, i.e. infinite dimensional generalizations of the classical Nambu bracket, or other bilinear forms, which reproduce the dynamics correctly when they act on a generalized helicity functional, instead of the Hamiltonian. Subsequently, a restriction of the dynamics to respect translational symmetry was imposed in order to retrieve the useful 4-field model of [53]. Although a proof of the existence or nonexistence of the corresponding Jacobi identities was not pursued, it is argued that these formulations are useful from a practical point of view for constructing conservative codes that avoid spurious dissipation of energy and helicity and also instabilities due to energy accumulation in unresolved scales.

The chapter closes with an attempt to use the energy-Casimir ingredients, to reconstruct the dynamics. For this purpose, the Hamiltonian and the Casimirs of the simple RMHD model were considered in order to derive a generic 2D hydromagnetic model that conserves the three ideal RMHD invariants by imposing a priori the RMHD conservation laws as orthogonality conditions. The dynamical equations are constructed in a heuristic way using the Jacobi-Poisson bracket as a building block. The Lie-Poisson and the Nambu brackets, for this generic model follow as simple consequences of the construction procedure. In addition, three families of hydromagnetic models that conserve any two out of the three RMHD invariants, were produced. It is proposed that some of these, or similar models, could be candidates for incorporating small length scale physics into the RMHD framework and as conservative regularizations of the RMHD system, preventing the flow from forming vorticity singularities in numerical simulations. Also, one can argue that some of these models could potentially be useful in regulating the ruggedness of the helicities and the energy individually by introducing small nondissipative terms. It is though to be proved if the proposed approach can potentially have other applications, and if some of the models presented here are indeed of physical relevance or practical importance.

6.2 Future prospects

The variety of methods and approaches employed in this study gives rise to new ideas on several potential future extensions, adaptations and developments. Some of these ideas can be classified in four main categories. The first concerns the physics contained in the models studied in this thesis, which can be enriched including additional physical mechanisms and considering additional possibilities that pertain to fusion and astrophysical plasmas. Second is the development of more versatile and robust numerical schemes for the equilibrium problem, which will allow for the introduction of additional contributions such as electron inertia and also for a more efficient representation of shaped domains. Third is the implementation of the stability criteria derived herein for practical applications and fourth, the development of conservative numerical schemes based on the trilinear bracket descriptions of Chapter 5.

More specifically, kinetic effects could be incorporated into the XMHD framework in future studies. For example hybrid fluid-kinetic Hamiltonian models have already been employed in the framework of MHD forming EC principles for planar plasmas [136] and deriving new stability criteria. The motivation is that the inclusion of kinetic effects is important for a multiscale description, especially when energetic particle populations exist within the plasma. Another possibility is the inclusion of anisotropic pressure effects which is justified by the difference of the thermal conductivities parallel and perpendicular to the magnetic field lines when $\mathbf{B}$ is strong, as is the case in Tokamaks.

Regarding equilibrium we can think of several new directions for obtaining novel extensions. For example the inclusion of electron inertial effects can be effected upon introducing into our numerical scheme the third flux function ξ and the corresponding Grad-Shafranov equation. A second extension could be the utilization of conformal mapping techniques in order to transform the physical domain in a computational domain of simpler shape, e.g. a disk. This will facilitate an accurate representation of the boundary without the need of considering dense grids which are computationally inconvenient. In addition, there are some particularly interesting recent developments in the computation of MHD equilibria using relaxation methods, such as the simulated annealing approach [137] and the relaxation via collision brackets [138]. Employing these new methods and introducing additionally two fluid effects seems particularly appealing.

Regarding stability, one can easily understand that the dynamically accessible and Lagrangian stability criteria found in this thesis are difficult to be applied on practical computations. One should resort to special cases, e.g. equilibria with purely toroidal flows, to find integrals that can simplify the criteria and lead to explicit sufficient stability conditions such as the MHD condition of reference [139]. Therefore, an interesting future extension could be the pursuit of special, simplified explicit criteria within the two fluid context or other approaches involving for example spectral analysis.

Finally, one of the most exciting prospects is the utilization of both structure preserving and simpler conservative numerical algorithms for the simulation of the nonlinear dynamics of models with electron inertia. For example, a variational integrator approach such as the one employed in [140] could also be employed for spatially reduced XMHD models for simulating collisionless reconnection mediated by both ion and electron dynamics. As a final point, the trilinear brackets computed in Chapter 5 could be employed for constructing conservative numerical algorithms that will preserve both the energy and an additional Casimir to high precision, thus improving the stability and the fidelity of simulations.

Appendix A

Outline of the numerical scheme of Chapter 3

The core of the solver that was constructed to compute barotropic HMHD equilibria with axial symmetry is a discretized version of Eq. (3.46). For this discretization, central finite difference approximations of the derivatives were employed yielding the following discrete form of (3.46)

$$\begin{aligned}(\Delta^*\psi)_{i,j} = \quad & \frac{\psi_{i+1,j} - 2\psi_{i,j} + \psi_{i-1,j}}{h_r^2} \\ & -\frac{1}{r_i}\frac{\psi_{i+1,j} - \psi_{i-1,j}}{2h_r} \\ & +\frac{\psi_{i,j+1} - 2\psi_{i,j} + \psi_{i,j-1}}{h_z^2} = rhs_\psi(r_i, \rho_{i,j}, \psi_{i,j}, \varphi_{i,j})\,, \end{aligned} \tag{A.1}$$

where h_r and h_z are the discretization lengths in the r and z direction, respectively and $rhs_\psi = -\mathcal{G}'(\mathcal{F} + \mathcal{G}) - \rho r^2 \mathcal{N}' - \rho(\varphi - \psi)/d_i^2$. The system of algebraic equations that results from (A.1) is solved iteratively using the SOR method, i.e. starting from an initial iterate, which is a known initial state[1], we compute the next iterate using the formula

$$\psi_{i,j}^{(n+1)} = \omega \Psi_{i,j} + (1-\omega)\psi_{i,j}^{(n)}\,, \tag{A.2}$$

where $0 < \omega < 2$ is the relaxation parameter and

$$\begin{aligned}\Psi_{i,j} = \frac{1}{C_{11}}\Big[& rhs_\psi(r_i, \rho_{i,j}^{(n)}, \psi_{i,j}^{(n)}, \varphi_{i,j}^{(n)}) - C_{21}\psi_{i+1,j}^{(n)} \\ & - C_{01}\psi_{i-1,j}^{(n+1)} - C_{12}\psi_{i,j+1}^{(n)} - C_{10}\psi_{i,j-1}^{(n+1)}\Big]\,, \end{aligned} \tag{A.3}$$

[1] In our case this is a static HMHD state computed previously also using the SOR algorithm. The initial condition for this static state was $\psi = 0$, $\rho = 0$.

with

$$\begin{aligned} C_{11} &= -2\frac{h_r^2+h_z^2}{h_r^2h_z^2}\,, & \text{(A.4)}\\ C_{12} &= C_{10} = \frac{1}{h_z^2}\,, & \text{(A.5)}\\ C_{21} &= \frac{1}{h_r^2} - \frac{1}{2r_ih_r}\,, & \text{(A.6)}\\ C_{21} &= \frac{1}{h_r^2} + \frac{1}{2r_ih_r}\,. & \text{(A.7)} \end{aligned}$$

We employ a red-black ordering which means that we have a "checkerboard" grid that is sweeped in two steps: in the first half-sweep we compute the red points with $mod(i+j,2) = 0$ and subsequently in the second half-sweep the black points with $mod(i+j,2) = 1$. The iteration is repeated until we have a good convergence below a predetermined small tolerance in the convergence rate and also in the residual error of ψ defined as

$$re_\psi = max((\Delta\psi^{(n+1)})_{i,j} - rhs_\psi(r_i, \rho_{i,j}^{(n+1)}, \psi_{i,j}^{(n+1)}, \varphi_{i,j}^{(n+1)}))\,,$$
$$i = 1, ..., N_r\,,\ j = 1, ..., N_z\,. \qquad \text{(A.8)}$$

The updated mass density $\rho^{(n+1)}$ is computed from the Bernoulli equation (3.48) using Brent's algorithm which is described below and $\varphi^{(n+1)}$ is computed upon solving the discrete version of (3.45) for $(\varphi - \psi)$, that is

$$\varphi_{i,j}^{(n+1)} = \psi_{i,j}^{(n+1)} + \frac{d_i^2}{\rho_{i,j}^{(n+1)}}\Big\{ \left[\mathcal{F}(\varphi^{(n)}) + \mathcal{G}(\psi^{(n+1)})\right]\mathcal{F}'(\varphi^{(n)}) + r^2\rho^{(n+1)}\mathcal{M}'(\varphi^{(n)})$$
$$-d_i^2 r^2 \mathcal{F}'(\varphi^{(n)})\nabla\cdot\left(\frac{\nabla\mathcal{F}(\varphi^{(n)})}{r^2\rho^{(n+1)}}\right)\Big\}_{i,j}\,. \qquad \text{(A.9)}$$

Within the D-shaped domain the solution is initialized with a static HMHD ($\varphi^{(0)} = \psi^{(0)}$) solution $\psi^{(0)}$ and a typical peaked-on-axis mass density $\rho^{(0)}$. The grid points lying inside the boundary are labeled with an $index = 1$ while those lying outside have $index = 0$. This assignment is done automatically upon comparing the distance of any grid point with the corresponding distance of the boundary curve from the geometric center. For $index = 0$ we set $\psi = \varphi = \rho = 0$ which are our boundary conditions and subsequently the iterations are updating only the values of the inner points.

Equilibrium solver pseudocode

input $\psi^{(0)}$ (initial iterate for ψ), $\rho^{(0)}$ (initial iterate for ρ), ω (relaxation parameter), N_r, N_z, tol (tolerance)
for $i = 1, ..., N_r$
for $j = 1, ..., N_z$
$\psi_{i,j} \longleftarrow 0$
$\varphi_{i,j} \longleftarrow 0$
$\rho_{i,j} \longleftarrow 0$
if $index = 1$ then
$\psi_{i,j} \longleftarrow \psi^{(0)}_{i,j}$
$\varphi_{i,j} \longleftarrow \psi^{(0)}_{i,j}$
$\rho_{i,j} \longleftarrow \rho^{(0)}_{i,j}$
end if
end for
end for
while $(cr > tol) \vee (re > tol)$ do
$\psi_{old} \longleftarrow \psi$
$isw \longleftarrow 1$
for $hsw = 1, 2$ (half-sweeps)
$jsw \longleftarrow isw$
for $i = 2, N_r - 1$
for $j = jsw + 1, N_z - 1, 2$
if $index = 1$ then
$\psi_{i,j} \longleftarrow \omega\Psi_{i,j} + (1-\omega)\psi_{i,j}$ (Ψ computed by (A.3))
$\rho_{i,j} \longleftarrow brent(\rho_{min}, \rho_{max}, b(\rho))$
$\varphi_{i,j} \longleftarrow$ rhs of (A.9)
end if
end for
$jsw \longleftarrow 3 - jsw$
end for
$isw \longleftarrow 3 - jsw$
adapt ω to accelerate convergence
end for
$cr \longleftarrow max(|\psi - \psi_{old}|)$
$re \longleftarrow max(|\Delta^*\psi - rhs_\psi|)$
end while

Brent's algorithm pseudocode

input a, b, f, tol ($a = \rho_{min}$, $b = \rho_{max}$, $f = b(\rho)$ i.e. the Bernoulli function)
test if $f(a)f(b) \leq 0$ otherwise exit
if $|f(a)| < |f(b)|$ then
$swap(a, b)$
end if
$c \longleftarrow a$
set mflag
while $|b - a| > tol$ do
if $f(a) \neq f(c)$ and $f(b) \neq f(c)$ then
$s \longleftarrow \frac{af(b)f(c)}{[f(a)-f(b)][f(a)-f(c)]} + \frac{bf(c)f(a)}{[f(b)-f(c)][f(b)-f(a)]} + \frac{cf(a)f(b)}{[f(c)-f(a)][f(c)-f(b)]}$
(inverse quadratic interpolation)
else
$s \longleftarrow b - f(b)\frac{b-a}{f(b)-f(a)}$ (secant)
end if
if ($s \notin [(3a + b)/4, b]$) $\vee$ (mflag set $\wedge$ ($|s - b| \geq |b - c|/2$)) $\vee$
$\vee$(mflag cleared $\wedge$ ($|s - b| \geq |c - d|/2$)) $\vee$
$\vee$ (mflag set $\wedge$ ($|b - c| < |\delta|$)) $\vee$ (mflag cleared $\wedge$ $|c - d| < |\delta|$) then
$s \longleftarrow \frac{a+b}{2}$ (bisection)
set mflag
else
clear mflag
end if
$d \longleftarrow c$
$c \longleftarrow b$
if $f(a)f(s) < 0$ then
$b \longleftarrow s$
else
$a \longleftarrow s$
end if
if $|f(a)| < |f(b)|$ then
$swap(a, b)$
end if
end while
return b

Appendix B

Direct derivation of incompressible, helically symmetric XMHD Grad-Shafranov equations

The direct derivation of the incompressible Grad-Shafranov system is effected upon projecting appropriately the XMHD stationary equations obtained from (1.27)– (1.28) with $\partial_t \mathbf{u} = 0$ and $\rho = 1$, i.e.

$$\mathbf{v} \times (\nabla \times \mathbf{v}) + (\nabla \times \mathbf{B}) \times \mathbf{B}^* = \nabla \tilde{p} \,, \tag{B.1}$$

$$[\mathbf{v} - d_i(\nabla \times \mathbf{B})] \times \mathbf{B}^* + d_e^2(\nabla \times \mathbf{B}) \times (\nabla \times \mathbf{v}) = \nabla \tilde{\Phi} \,, \tag{B.2}$$

where $\tilde{p} := p + |\mathbf{v}|^2/2 + d_e^2|\mathbf{J}|^2/2$ and $\tilde{\Phi} := \Phi - p_e$. From the helical representation for the fields (2.6)–(2.7) and their curls (2.9)–(2.10) we find that the generalized vorticities

$$\mathbf{B}^\gamma = \mathbf{B}^* + \gamma \nabla \times \mathbf{v} \,, \tag{B.3}$$

$$\mathbf{B}^\mu = \mathbf{B}^* + \mu \nabla \times \mathbf{v} \,, \tag{B.4}$$

can be decomposed as

$$\mathbf{B}^\gamma = k^{-1} B_h^\gamma \mathbf{h} + \nabla \varphi \times \mathbf{h} \,, \tag{B.5}$$

$$\mathbf{B}^\mu = k^{-1} B_h^\mu \mathbf{h} + \nabla \xi \times \mathbf{h} \tag{B.6}$$

with

$$B_h^\gamma = B_h^* + \gamma k^{-1} \mathcal{L}\chi - 2n\ell\gamma k^2 v_h \,, \tag{B.7}$$

$$B_h^\mu = B_h^* + \mu k^{-1} \mathcal{L}\chi - 2n\ell\mu k^2 v_h \,, \tag{B.8}$$

$$\varphi = \psi^* + \gamma k^{-1} v_h \,, \tag{B.9}$$

$$\xi = \psi^* + \mu k^{-1} v_h \,. \tag{B.10}$$

Performing the operation $\gamma((\mathrm{B}.1)) + (\mathrm{B}.2)$ we find

$$\mathbf{v}\times\mathbf{B}^\gamma + (\gamma - d_i)\mathbf{J}\times\mathbf{B}^* + d_e^2\mathbf{J}\times(\nabla\times\mathbf{v}) = \nabla f\,, \tag{B.11}$$

where $f := \tilde{\Phi}+\gamma\tilde{p}$. Projecting the equation above along $\mathbf{B}^\gamma$ and exploiting the relation $d_e^2 = \gamma(\gamma - d_i)$ we find $\mathbf{B}^\gamma\cdot\nabla f = 0$ which implies

$$\tilde{\Phi} + \gamma\tilde{p} = f(\varphi)\,. \tag{B.12}$$

Similarly, forming $\mu((\mathrm{B}.1)) + (\mathrm{B}.2)$ and projecting along $\mathbf{B}^\mu$ we find

$$\tilde{\Phi} + \mu\tilde{p} = g(\xi)\,. \tag{B.13}$$

Note that f and g represent arbitrary functions of φ and ξ, respectively. In view of the above results we can write $\gamma((\mathrm{B}.1)) + (\mathrm{B}.2)$ and $\mu((\mathrm{B}.1)) + (\mathrm{B}.2)$ as

$$(\mathbf{v} - \mu\mathbf{J})\times\mathbf{B}^\gamma = f'\nabla\varphi\,, \tag{B.14}$$

$$(\mathbf{v} - \gamma\mathbf{J})\times\mathbf{B}^\mu = g'\nabla\xi\,, \tag{B.15}$$

where $\gamma - d_i = -\mu$ and $\mu - d_i = -\gamma$ have been used. Inserting the helically symmetric fields (2.6)–(2.7) into (B.14) results in

$$k(v_h - \mu J_h)\nabla\varphi - kB_h^\gamma\nabla(\chi - \mu k^{-1}B_h) + [\chi - \mu k^{-1}B_h, \varphi]\mathbf{h} = f'\nabla\varphi\,, \tag{B.16}$$

where $[a, b] = (\nabla a\times\nabla b)\cdot\mathbf{h}$. Projecting (B.16) along $\mathbf{h}$ yields

$$\chi - \mu k^{-1}B_h = K(\varphi)\,. \tag{B.17}$$

Using this result, after projecting (B.16) along $\nabla\varphi$ we find

$$k(v_h - \mu J_h) - kB_h^\gamma K'(\varphi) = f'(\varphi)\,. \tag{B.18}$$

Following a similar procedure for (B.15) the following relations can be derived

$$\chi - \gamma k^{-1}B_h = \Lambda(\xi)\,, \tag{B.19}$$

$$k(v_h - \gamma J_h) - kB_h^\mu\Lambda'(\xi) = g'(\xi)\,. \tag{B.20}$$

From Eqs. (B.17) and (B.19) one can readily find that

$$B_h = k\frac{K(\varphi) - \Lambda(\xi)}{\gamma - \mu}\,, \tag{B.21}$$

$$\chi = \frac{\gamma K(\varphi) - \mu\Lambda(\xi)}{\gamma - \mu}\,. \tag{B.22}$$

Substituting now $J_h = k^{-1}\mathcal{L}\psi - 2n\ell k^2 B_h$ (see Chapter 2) and (B.7) in (B.18) yields

$$kv_h - \mu k(k^{-1}\mathcal{L}\psi - 2n\ell k^2 B_h) - k(B_h^* + \gamma k^{-1}\mathcal{L}\chi - 2n\ell\gamma k^2 v_h)K'(\varphi) = f'(\varphi)\,. \quad \text{(B.23)}$$

Upon defining

$$\mathcal{F}(\varphi) := \frac{K(\varphi)}{\gamma-\mu}\,, \quad \mathcal{G}(\xi) := \frac{\Lambda(\xi)}{\mu-\gamma}\,, \quad \text{(B.24)}$$

$$\mathcal{M}(\varphi) := \frac{f(\varphi)}{\gamma-\mu}\,, \quad \mathcal{N}(\xi) := \frac{g(\xi)}{\mu-\gamma}\,, \quad \text{(B.25)}$$

we have

$$B_h = k(\mathcal{F}+\mathcal{G})\,, \quad \chi = \gamma\mathcal{F} + \mu\mathcal{G}\,. \quad \text{(B.26)}$$

In view of (2.35) and using (B.26), after some algebra Eq. (B.23) becomes

$$(\gamma^2+d_e^2)\mathcal{F}'\mathcal{L}\mathcal{F} = -(1+\varsigma)k^2\mathcal{F}'(\mathcal{F}+\mathcal{G}) - \mathcal{M}' - \left(\frac{\mu}{\gamma-\mu} - 2n\ell d_e^2 k^2\mathcal{F}'\right)\mathcal{L}\psi$$
$$+2n\ell\frac{\mu}{\gamma-\mu}k^4(\mathcal{F}+\mathcal{G}) + k^2\left[\frac{1}{(\gamma-\mu)^2} - \frac{2n\ell\gamma k^2}{\gamma-\mu}\mathcal{F}'\right](\varphi-\xi)\,. \quad \text{(B.27)}$$

Similarly, from (B.20) we can arrive at

$$(\mu^2+d_e^2)\mathcal{G}'\mathcal{L}\mathcal{G} = -(1+\varsigma)k^2\mathcal{G}'(\mathcal{F}+\mathcal{G}) - \mathcal{N}' + \left(\frac{\gamma}{\gamma-\mu} + 2n\ell d_e^2 k^2\mathcal{G}'\right)\mathcal{L}\psi$$
$$-2n\ell\frac{\gamma}{\gamma-\mu}k^4(\mathcal{F}+\mathcal{G}) - k^2\left[\frac{1}{(\gamma-\mu)^2} - \frac{2n\ell\mu k^2}{\gamma-\mu}\mathcal{G}'\right](\varphi-\xi)\,. \quad \text{(B.28)}$$

For the last calculations we have used (3.29) and $\gamma\mu = -d_e^2$. The equations above are simply Eqs. (3.33)–(3.34) with $\rho = 1$, as expected. Regarding the Bernoulli equation, from

$$(\gamma-\mu)\mathcal{M}(\varphi) = \tilde{\Phi} + \gamma\tilde{p}\,, \quad (\mu-\gamma)\mathcal{N}(\xi) = \tilde{\Phi} + \mu\tilde{p}\,, \quad \text{(B.29)}$$

we take

$$\tilde{p} = \mathcal{M}(\varphi) + \mathcal{N}(\xi)\,, \quad or \quad \text{(B.30)}$$

$$p = \mathcal{M}(\varphi) + \mathcal{N}(\xi) - \frac{|\mathbf{v}|^2}{2} - \frac{d_e^2}{2}|\mathbf{J}|^2\,, \quad \text{(B.31)}$$

which is Eq. (3.60).

Appendix C

Additional proofs for Chapter 4

C.1 XMHD equilibria with toroidal rotation

Let us consider equilibria with purely toroidal rotation. To find the equilbrium conditions let us consider Eqs. (1.27), (1.29) with $\partial_t \to 0$ and $\mathbf{v} = rv_\phi \nabla\phi$. In this case the XMHD equations reduce to

$$
\begin{aligned}
&r^{-1}v_\phi \nabla(rv_\phi) - \nabla\left(h + \frac{|\mathbf{v}|^2}{2} + d_e^2\frac{|\mathbf{J}|^2}{2\rho^2}\right) \\
&-\rho^{-1}\left[\frac{\Delta^*\psi}{r^2}\nabla\psi^* + \frac{B_\phi^*}{r}\nabla(rB_\phi) - \nabla(rB_\phi)\cdot(\nabla\psi^*\times\nabla\phi)\nabla\phi\right] = 0\,, \qquad \text{(C.1)}
\end{aligned}
$$

$$
\begin{aligned}
&r^{-1}v_\phi\nabla\psi^* - \rho^{-1}\Big\{d_i\left[-\frac{\Delta^*\psi}{r^2}\nabla\psi^* - \frac{B_\phi^*}{r}\nabla(rB_\phi) + \nabla(rB_\phi)\cdot(\nabla\psi^*\times\nabla\phi)\nabla\phi\right] \\
&+d_e^2\left[\frac{\Delta^*\psi}{r^2}\nabla(rv_\phi) - \nabla(rB_\phi)\cdot(\nabla(rv_\phi)\times\nabla\phi)\nabla\phi\right]\Big\} = \nabla\tilde{\Phi}\,, \qquad \text{(C.2)}
\end{aligned}
$$

where $\tilde{\Phi} = \Phi - d_i h_e + d_e^2\rho^{-1}\mathbf{v}\cdot\mathbf{J} - d_i d_e^2\rho^{-2}|\mathbf{J}|^2$, with Φ and h_e being the equilibrium electrostatic potential and electron specific enthalpy, respectively. Projecting Eq. (C.1) along $\nabla\phi$ we find

$$
\nabla(rB_\phi)\cdot(\nabla\psi^*\times\nabla\phi) = 0\,, \Leftrightarrow rB_\phi = F(\psi^*)\,. \qquad \text{(C.3)}
$$

Projecting Eq. (C.2) along $\nabla\phi$ and using result (C.3) we find

$$
\nabla(rv_\phi)\cdot(\nabla\psi^*\times\nabla\phi) = 0\,, \Leftrightarrow rv_\phi = G(\psi^*)\,. \qquad \text{(C.4)}
$$

Eqs. (C.3) and (C.4) imply $\mathbf{J} = -\Delta^*\psi\nabla\phi + F'(\psi^*)\nabla\psi^*\times\nabla\phi$ and $\boldsymbol{\omega} = G'(\psi^*)\nabla\psi^*\times\nabla\phi$, respectively. Therefore $\mathbf{J}\cdot\nabla\psi^* = \boldsymbol{\omega}\cdot\nabla\psi^* = 0$. This means that all three vector fields $\mathbf{v}$, $\mathbf{B}^*$ and $\mathbf{J}$ lie on common flux surfaces labeled by ψ^*. This property of common flux surfaces, was crucial for the derivation of a sufficient stability criterion in the context of MHD [139] for a three-dimensional incompressible displacement vector field. It is thus interesting to pursue the investigation of this possibility also in the context of XMHD in the future. As regards the current application, we confine the perturbation vectors to be tangent to the characteristic surfaces. Also note that using result (C.3) and projecting (C.1) along $\mathbf{B}^*$ we find

$$\nabla\tilde{h}\cdot(\nabla\psi^*\times\nabla\phi)=0\,,\Leftrightarrow\tilde{h}=\tilde{h}(\psi^*)\,. \tag{C.5}$$

For equilibria with purely toroidal flows, subject to perturbations with displacement vectors tangent to the common surfaces, it easy to understand that every product of the form $\mathbf{b}_i\times\mathbf{c}_j$ where $\mathbf{b}=(\boldsymbol{\zeta},\boldsymbol{\eta})$ and $\mathbf{c}=(\mathbf{v},\mathbf{B}^*,\mathbf{J})$, will be parallel to the vector $\nabla\psi^*$ at each surface point, i.e. $\mathbf{b}_i\times\mathbf{c}_j=g_{ij}(r,z)\nabla\psi^*$. Therefore every vector of the form $\nabla\times(\mathbf{b}\times\mathbf{c})$ will be $\nabla g\times\nabla\psi^*$ and consequently every term $(\mathbf{b}_i\times\mathbf{c}_j)\cdot\nabla\times(\mathbf{b}_k\times\mathbf{c}_\ell)$ in (4.72) will vanish. The same holds also for terms containing $\mathbf{b}_i\cdot(\mathbf{c}_j\times\mathbf{c}_k)$, since $(\mathbf{c}_j\times\mathbf{c}_k)$ is normal to the characteristic surfaces at each point, if not zero. In addition the term containing $\boldsymbol{\zeta}\cdot\nabla\tilde{h}$ will vanish as well due to (C.5). A rigorous proof can be carried out upon writing

$$\boldsymbol{\zeta}=r\zeta_\phi\nabla\phi+\frac{(\boldsymbol{\zeta}\cdot\mathbf{B}_p^*)}{|\mathbf{B}_p^*|^2}\mathbf{B}_p^*\,, \tag{C.6}$$

which is a general representation of vectors tangent to the surfaces $\psi^*=const.$, similarly for $\boldsymbol{\eta}$ and computing every single term in (4.72), leading eventually to (4.76).

C.2 Proof of relation (4.78)

To prove Eq. (4.78), one may start from the lhs and try to prove that it equals to the rhs, upon performing some vector analysis manipulations and also using the divergence theorem as follows

$$\begin{aligned}
&\int d^3x\,\rho(\boldsymbol{\zeta}\times\mathbf{b})\cdot(\mathbf{a}\cdot\nabla\boldsymbol{\eta}-\boldsymbol{\eta}\cdot\nabla\mathbf{a})\\
&=\int d^3x\,\rho\left\{\mathbf{a}\cdot\nabla\left[\boldsymbol{\eta}\cdot(\boldsymbol{\zeta}\times\mathbf{b})\right]-\boldsymbol{\eta}\cdot(\mathbf{a}\cdot\nabla)(\boldsymbol{\zeta}\times\mathbf{b})-(\boldsymbol{\zeta}\times\mathbf{b})\cdot(\boldsymbol{\eta}\cdot\nabla\mathbf{a})\right\}\\
&=-\int d^3x\,\big\{\boldsymbol{\eta}\cdot(\boldsymbol{\zeta}\times\mathbf{b})\nabla\cdot(\rho\mathbf{a})+\rho\left[(\mathbf{b}\times\boldsymbol{\eta})\cdot(\mathbf{a}\cdot\nabla\boldsymbol{\zeta})+(\boldsymbol{\eta}\times\boldsymbol{\zeta})\cdot(\mathbf{a}\cdot\nabla\mathbf{b})\right]\\
&\qquad+(\boldsymbol{\zeta}\times\mathbf{b})\cdot(\boldsymbol{\eta}\cdot\nabla\mathbf{a})\big\}\,,
\end{aligned} \tag{C.7}$$

where the last expression was obtained upon integrating by parts and omitting the resulting surface integral. Now one can use the following vector formula

$$\begin{aligned}
&(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\cdot\nabla\mathbf{d})+(\mathbf{c}\times\mathbf{a})\cdot(\mathbf{b}\cdot\nabla\mathbf{d})\\
&+(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{a}\cdot\nabla\mathbf{d})=(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}(\nabla\cdot\mathbf{d})\,,
\end{aligned} \tag{C.8}$$

to write the last term in (C.7) as

$$-\int d^3x\,\rho(\boldsymbol{\zeta}\times\mathbf{b})\cdot(\boldsymbol{\eta}\cdot\nabla\mathbf{a}) = -\int d^3x\,\rho\big[(\boldsymbol{\zeta}\times\mathbf{b})\cdot\boldsymbol{\eta}(\nabla\cdot\mathbf{a}) \\ -(\boldsymbol{\eta}\times\boldsymbol{\zeta})\cdot(\mathbf{b}\cdot\nabla\mathbf{a}) - (\mathbf{b}\times\boldsymbol{\eta})\cdot(\boldsymbol{\zeta}\cdot\nabla\mathbf{a})\big]\,, \tag{C.9}$$

therefore one has

$$\int d^3x\,\rho(\boldsymbol{\zeta}\times\mathbf{b})\cdot(\mathbf{a}\cdot\nabla\boldsymbol{\eta}-\boldsymbol{\eta}\cdot\nabla\mathbf{a}) = \int d^3x\,\rho(\boldsymbol{\eta}\times\mathbf{b})\cdot(\mathbf{a}\cdot\nabla\boldsymbol{\zeta}-\boldsymbol{\zeta}\cdot\nabla\mathbf{a}) \\ -\int d^3x\,(\boldsymbol{\eta}\times\boldsymbol{\zeta})\cdot[\mathbf{b}\nabla\cdot(\rho\mathbf{a}) + \rho(\mathbf{b}\nabla\cdot\mathbf{a} + \mathbf{a}\cdot\nabla\mathbf{b} - \mathbf{b}\cdot\nabla\mathbf{a})]\,. \tag{C.10}$$

Using the following vector identity

$$\nabla\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}\nabla\cdot\mathbf{b} - \mathbf{b}\nabla\cdot\mathbf{a} + \mathbf{b}\cdot\nabla\mathbf{a} - \mathbf{a}\cdot\nabla\mathbf{b} \tag{C.11}$$

(C.10) takes the form (4.78).

Appendix D

Energy-Casimir stability of planar XMHD equilibria

A simple case for which the Energy-Casimir stability analysis works without the need of making a posteriori restrictive assumptions, is obtained upon considering a planar XMHD model. That is, the plasma motion takes place on a plane perpendicular to a magnetic field with straight field lines. Therefore, the appropriate field representation is given by

$$\mathbf{B} = B_z\hat{z}\,, \tag{D.1}$$

$$\mathbf{v} = \nabla\chi\times\hat{z}+\nabla\Upsilon\,, \tag{D.2}$$

$$B_z^* = B_z - d_e^2\nabla\cdot\left(\frac{\nabla B_z}{\rho}\right)\,. \tag{D.3}$$

The corresponding Poisson bracket is given by bracket (39) of [71] with $\psi^*=v_z=F_{v_z}=F_{\psi^*}=0$, i.e.

$$\begin{aligned}\{F,G\}_{planar} = \int d^2x\big\{F_\rho\Delta G_w - G_\rho\Delta F_w + \rho^{-1}\Omega\big([F_\Omega,G_\Omega]+[F_w,G_w]\\ +\nabla F_w\cdot\nabla G_\Omega - \nabla F_\Omega\cdot\nabla G_w\big) + \rho^{-1}B_z^*\big([F_\Omega,G_{B_z^*}]-[G_\Omega,F_{B_z^*}]\\ +\nabla F_w\cdot\nabla G_{B_z^*} - \nabla G_w\cdot\nabla F_{B_z^*}\big) - d_i\rho^{-1}B_z^*[F_{B_z^*},G_{B_z^*}] + d_e^2\rho^{-1}\Omega[F_{B_z^*},G_{B_z^*}]\big\}\,,\end{aligned} \tag{D.4}$$

where $\Omega=-\Delta\chi$ and $w=\Delta\Upsilon$. The Hamiltonian reads as follows

$$\mathcal{H} = \frac{1}{2}\int d^2x\left(\rho|\nabla\chi|^2+\rho|\nabla\Upsilon|^2+2\rho[\Upsilon,\chi]+2\rho U(\rho)+B_z^*B_z\right)\,. \tag{D.5}$$

From the Poisson bracket (D.4) we can find that planar XMHD possess the following two infinite families of Casimir invariants

$$\mathcal{C}_\pm = \int d^2x\rho\mathcal{F}_\pm\left(\frac{B_z^\pm}{\rho}\right)\,, \tag{D.6}$$

where $B_z^{\pm} := B_z^* + \gamma_{\pm}\Omega$ with $\gamma_{\pm} = (d_i \pm \sqrt{d_i^2 + 4d_e^2})/2$. Applying the Energy-Casimir variational principle we have

$$\int d^2x \Bigg\{ \left[h(\rho) + \frac{v^2}{2} - \sum_{\pm} \left[\mathcal{F}_{\pm} - \rho^{-1} B_z^{\pm} \mathcal{F}'_{\pm} \right] + \frac{d_e^2}{2\rho^2} |\nabla B_z|^2 \right] \delta\rho$$

$$+ \left(B_z - \sum_{\pm} \mathcal{F}'_{\pm} \right) \delta B_z^* + \left(\rho \mathbf{v} - \sum_{\pm} \gamma_{\pm} \nabla \mathcal{F}'_{\pm} \times \hat{z} \right) \cdot \delta \mathbf{v} \Bigg\} = 0 \,, \qquad \text{(D.7)}$$

from which the following equilibrium equations are deduced

$$h(\rho) = \sum_{\pm} \left[\mathcal{F}_{\pm} - \rho^{-1} B_z^{\pm} \mathcal{F}'_{\pm} \right] - \frac{v^2}{2} - \frac{d_e^2}{2\rho^2} |\nabla B_z|^2 \,, \qquad \text{(D.8)}$$

$$B_z = \sum_{\pm} \mathcal{F}'_{\pm} \,, \qquad \mathbf{v} = \rho^{-1} \sum_{\pm} \gamma_{\pm} \nabla \mathcal{F}'_{\pm} \times \hat{z} \,. \qquad \text{(D.9)}$$

Let us now take the second variation of the EC functional

$$\delta^2 \mathcal{H}_C = \int d^2x \Bigg\{ \left[h'(\rho) - \frac{d_e^2}{\rho^3} |\nabla B_z|^2 \right] (\delta\rho)^2 + 2\mathbf{v} \cdot \delta\mathbf{v} \delta\rho$$

$$+ \rho |\delta \mathbf{v}|^2 + (\delta B_z)^2 + \frac{d_e^2}{\rho} |\nabla \delta B_z|^2 - \rho \sum_{\pm} \mathcal{F}''_{\pm} \left[\delta \left(\frac{B_z^{\pm}}{\rho} \right) \right]^2 \Bigg\} \,, \qquad \text{(D.10)}$$

where we have used the definitions for B_z^* and $B_z^{\pm}$. Upon completing squares we find

$$\delta^2 \mathcal{H}_C = \int d^2x \Bigg\{ \rho^{-1} \left[c_s^2 - |\mathbf{v}|^2 - \frac{d_e^2}{\rho^2} |\nabla B_z|^2 \right] (\delta\rho)^2 + \rho |\delta \mathbf{v} + \rho^{-1} \mathbf{v} \delta\rho|^2$$

$$+ (\delta B_z)^2 + \frac{d_e^2}{\rho} |\nabla \delta B_z|^2 - \rho \sum_{\pm} \mathcal{F}''_{\pm} \left[\delta \left(\frac{B_z^{\pm}}{\rho} \right) \right]^2 \Bigg\} \,. \qquad \text{(D.11)}$$

Then for the quandratic form $\delta^2 \mathcal{H}_C$ to be positive definite we have the following sufficient conditions

$$|\mathbf{v}|^2 + \frac{d_e^2}{\rho^2} |\nabla B_z|^2 < c_s^2 \,, \qquad \mathcal{F}''_{\pm} < 0 \,. \qquad \text{(D.12)}$$

Bibliography

[1] J. D. Lawson. “Some Criteria for a Power Producing Thermonuclear Reactor”. In: *Proceedings of the Physical Society. Section B* 70.1 (1957), pp. 6–10. DOI: 10.1088/0370-1301/70/1/303.

[2] F. Wagner. “A quarter-century of H-mode studies”. In: *Plasma Physics and Controlled Fusion* 49.12B (2007), B1–B33. DOI: 10.1088/0741-3335/49/12b/s01.

[3] J. P. Freidberg. “Ideal magnetohydrodynamic theory of magnetic fusion systems”. In: *Rev. Mod. Phys.* 54 (3 1982), pp. 801–902. DOI: 10.1103/RevModPhys.54.801.

[4] H. Alfvén. “Existence of Electromagnetic-Hydrodynamic Waves”. In: *Nature* 150 (3805 1942), pp. 405–406. DOI: 10.1103/RevModPhys.54.801.

[5] D. D. Schnack. *Lectures in Magnetohydrodynamics: With an Appendix on Extended MHD*. Lecture Notes in Physics. Springer Berlin Heidelberg, 2009. ISBN: 9783642006876. DOI: 10.1007/978-3-642-00688-3.

[6] Reimar Lüst. “Über die Ausbreitung von Wellen in einem Plasma”. In: *Fortschr. Phys.* 7.9 (1959), pp. 503–558. DOI: 10.1002/prop.19590070902.

[7] H. Goldstein, C. P. Poole, and J. sL. Safko. *Classical Mechanics*. Addison Wesley, 2002. ISBN: 9780201657029.

[8] P. J. Morrison. “Hamiltonian description of the ideal fluid”. In: *Rev. Mod. Phys.* 70 (2 1998), pp. 467–521. DOI: 10.1103/RevModPhys.70.467.

[9] Z. Yoshida, P. J. Morrison, and F. J. Dobarro. “Singular Casimir Elements of the Euler Equation and Equilibrium Points”. In: *J. Math. Fluid Mech.* 16.1 (2014), pp. 41–57. DOI: https://doi.org/10.1007/s00021-013-0143-4.

[10] D. D. Holm et al. “Nonlinear stability of fluid and plasma equilibria”. In: *Phys. Rep.* 123.1 (1985), pp. 1–116. DOI: https://doi.org/10.1016/0370-1573(85)90028-6.

[11] V. I. Arnold. “A Variational principle for three-dimensional steady flows of an ideal fluid”. In: *Vladimir I. Arnold - Collected Works*. Ed. by Alexander B. Givental et al. Springer Berlin Heidelberg, 2014, pp. 1–7. ISBN: 978-3-642-31031-7. DOI: 10.1007/978-3-642-31031-7_1.

[12] P. J. Morrison and D. Pfirsch. “Free-energy expressions for Vlasov equilibria”. In: *Phys. Rev. A* 40 (7 1989), pp. 3898–3910. DOI: 10.1103/PhysRevA.40.3898.

[13] E. Hameiri. “Dynamically accessible perturbations and magnetohydrodynamic stability”. In: *Physics of Plasmas* 10.7 (2003), pp. 2643–2648. DOI: 10.1063/1.1577122.

[14] E. Hameiri and R. Torasso. “Linear stability of static equilibrium states in the Hall-magnetohydrodynamics model”. In: *Physics of Plasmas* 11.11 (2004), pp. 4934–4945. DOI: 10.1063/1.1784453.

[15] M. Hirota, Z. Yoshida, and E. Hameiri. “Variational principle for linear stability of flowing plasmas in Hall magnetohydrodynamics”. In: *Physics of Plasmas* 13.2 (2006), p. 022107. DOI: 10.1063/1.2169734.

[16] P. J. Morrison. “On Hamiltonian and Action Principle Formulations of Plasma Dynamics”. In: *AIP Conference Proceedings* 1188.1 (2009), pp. 329–344. DOI: 10.1063/1.3266810.

[17] P. J. Morrison, M. Lingam, and R. Acevedo. “Hamiltonian and action formalisms for two-dimensional gyroviscous magnetohydrodynamics”. In: *Phys. Plasmas* 21.8 (2014), p. 082102. DOI: 10.1063/1.4891321.

[18] H. M. Abdelhamid, Y. Kawazura, and Z. Yoshida. “Hamiltonian formalism of extended magnetohydrodynamics”. In: *J. Phys. A: Math. Theor.* 48.23 (2015), p. 235502. DOI: https://doi.org/10.1088/1751-8113/48/23/235502.

[19] P. J. Morrison. “Poisson brackets for fluids and plasmas”. In: *AIP Conference Proceedings* 88.1 (1982), pp. 13–46. DOI: 10.1063/1.33633.

[20] Eric C. D’Avignon, P. J. Morrison, and M. Lingam. “Derivation of the Hall and extended magnetohydrodynamics brackets”. In: *Physics of Plasmas* 23.6 (2016), p. 062101. DOI: 10.1063/1.4952641.

[21] M. Lingam, P. J. Morrison, and G. Miloshevich. “Remarkable connections between extended magnetohydrodynamics models”. In: *Phys. Plasmas* 22.7 (2015), p. 072111. DOI: 10.1063/1.4926821.

[22] I. B. Bernstein et al. “An energy principle for hydromagnetic stability problems”. In: *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences* 244.1236 (1958), pp. 17–40. DOI: 10.1098/rspa.1958.0023.

[23] E. Frieman and M. Rotenberg. “On Hydromagnetic Stability of Stationary Equilibria”. In: *Rev. Mod. Phys.* 32 (4 1960), pp. 898–902. DOI: 10.1103/RevModPhys.32.898.

[24] P. W. Terry. “Suppression of turbulence and transport by sheared flow”. In: *Rev. Mod. Phys.* 72 (1 2000), pp. 109–165. DOI: 10.1103/RevModPhys.72.109.

[25] C. Wahlberg and A. Bondeson. "Stabilization of the internal kink mode in a tokamak by toroidal plasma rotation". In: *Physics of Plasmas* 7.3 (2000), pp. 923–930. DOI: 10.1063/1.873889.

[26] R. L. Miller et al. "Stabilization of ballooning modes with sheared toroidal rotation". In: *Physics of Plasmas* 2.10 (1995), pp. 3676–3684. DOI: 10.1063/1.871067.

[27] M. S. Chu. "Shear flow destabilization of a slowly rotating tokamak". In: *Physics of Plasmas* 5.1 (1998), pp. 183–191. DOI: 10.1063/1.872687.

[28] I. T. Chapman et al. "The effects of sheared toroidal rotation on stability limits in tokamak plasmas". In: *Plasma Physics and Controlled Fusion* 53.12 (2011), p. 125002. DOI: 10.1088/0741-3335/53/12/125002.

[29] D. Brunetti, E. Lazzaro, and S. Nowak. "Ideal and resistive magnetohydrodynamic instabilities in cylindrical geometry with a sheared flow along the axis". In: *Plasma Physics and Controlled Fusion* 59.5 (2017), p. 055012. DOI: 10.1088/1361-6587/aa6423.

[30] X. L. Chen and P. J. Morrison. "A sufficient condition for the ideal instability of shear flow with parallel magnetic field". In: *Physics of Fluids B: Plasma Physics* 3.4 (1991), pp. 863–865. DOI: 10.1063/1.859841.

[31] V. V. Mishin and V. M. Tomozov. "Kelvin–Helmholtz Instability in the Solar Atmosphere, Solar Wind and Geomagnetosphere". In: *Solar Physics* 291.11 (2016), pp. 3165–3184. DOI: 10.1007/s11207-016-0891-4.

[32] S. A. Balbus and J. F. Hawley. "A powerful local shear instability in weakly magnetized disks. I - Linear analysis. II - Nonlinear evolution". In: *Astrophysical Journal* 376 (), p. 214. DOI: 10.1086/170270.

[33] A. Ishida, H. Momota, and L. C. Steinhauer. "Variational formulation for a multifluid flowing plasma with application to the internal tilt mode of a field-reversed configuration". In: *The Physics of Fluids* 31.10 (1988), pp. 3024–3034. DOI: 10.1063/1.866959.

[34] D. C. Barnes. "Stability of long field-reversed configurations". In: *Physics of Plasmas* 9.2 (2002), pp. 560–568. DOI: 10.1063/1.1435425.

[35] J. Birn et al. "Geospace Environmental Modeling (GEM) magnetic reconnection challenge". In: *J. Geophys. Res.* 106 (Mar. 2001), pp. 3715–3720. DOI: 10.1029/1999JA900449.

[36] N. Andrés, P. Dmitruk, and D. Gómez. "Influence of the Hall effect and electron inertia in collisionless magnetic reconnection". In: *Physics of Plasmas* 23.2 (2016), p. 022903. DOI: 10.1063/1.4942418.

[37] P. A. Gourdain. "The impact of the Hall term on tokamak plasmas". In: *arXiv* (2017), p. 1703.00987v2. URL: https://arxiv.org/abs/1703.00987.

[38] P. N. Guzdar, S. M. Mahajan, and Z. Yoshida. "A theory for the pressure pedestal in high (H) mode tokamak discharges". In: *Physics of Plasmas* 12.3 (2005), p. 032502. DOI: 10.1063/1.1852468.

[39] D. D. Holm. "Hall magnetohydrodynamics: Conservation laws and Lyapunov stability". In: *Phys. Fluids* 30.5 (1987), pp. 1310–1322. DOI: 10.1063/1.866246.

[40] V. I. Ilgisonis. "Variational Stability Problem for MHD Model with Hall Effect". In: *Fusion Technology* 35.1T (1999), pp. 170–174. DOI: 10.13182/FST99-A11963845.

[41] R. Torasso and E. Hameiri. "Ballooning mode stability in the Hall Magnetohydrodynamics model". In: *Physics of Plasmas* 12.3 (2005), p. 032106. DOI: 10.1063/1.1851993.

[42] K. Elsässer and P. Spiess. "Lyapunov stability of perfect fluid and multifluid plasma equilibria". In: *Physics Letters A* 230.1 (1997), pp. 67 –72. DOI: https://doi.org/10.1016/S0375-9601(97)00220-X.

[43] P. Spiess and K. Elsässer. "Two variational principles for incompressible plasmas". In: *Physics of Plasmas* 6.11 (1999), pp. 4208–4214. DOI: 10.1063/1.873687.

[44] P. J. Morrison and J. M. Greene. "Noncanonical Hamiltonian Density Formulation of Hydrodynamics and Ideal Magnetohydrodynamics". In: *Phys. Rev. Lett.* 45 (10 1980), pp. 790–794. DOI: 10.1103/PhysRevLett.45.790.

[45] L. Woltjer. "On hydromagnetic equilibrium". In: *Proceedings of the National Academy of Sciences* 44.9 (1958), pp. 833–841. DOI: 10.1073/pnas.44.9.833.

[46] L. Woltjer. "Hydromagnetic Equilibrium. III. Axisymmetric Incompressible Media." In: *ApJ* 130 (Sept. 1959), p. 400. DOI: 10.1086/146731.

[47] J. B. Taylor. "Relaxation of Toroidal Plasma and Generation of Reverse Magnetic Fields". In: *Phys. Rev. Lett.* 33 (19 1974), pp. 1139–1141. DOI: 10.1103/PhysRevLett.33.1139.

[48] M. D. Kruskal and C. R. Oberman. "On the Stability of Plasma in Static Equilibrium". In: *Phys. Fluids* 1.4 (1958), pp. 275–280. DOI: 10.1063/1.1705885.

[49] T. Andreussi, P. J. Morrison, and F. Pegoraro. "Hamiltonian magnetohydrodynamics: Lagrangian, Eulerian, and dynamically accessible stability–Theory". In: *Phys. Plasmas* 20.9 (2013), p. 092104. DOI: 10.1063/1.4819779.

[50] K. Kimura and P. J. Morrison. "On energy conservation in extended magnetohydrodynamics". In: *Phys. Plasmas* 21.8 (2014), p. 082101. DOI: 10.1063/1.4890955.

[51] M. J. Lighthill. "Studies on Magneto-Hydrodynamic Waves and other Anisotropic Wave Motions". In: *Philos. Trans. R. Soc. London A* 252.1014 (1960), pp. 397–430. DOI: `https://doi.org/10.1098/rsta.1960.0010`.

[52] M. Lingam, P. J. Morrison, and E. Tassi. "Inertial magnetohydrodynamics". In: *Phys. Lett. A* 379.6 (2015), pp. 570 –576. DOI: `https://doi.org/10.1016/j.physleta.2014.12.008`.

[53] D. Grasso et al. "Structure and computation of two-dimensional incompressible extended MHD". In: *Phys. Plasmas* 24.1 (2017), p. 012110. DOI: `10.1063/1.4974039`.

[54] A. Weller et al. "Persistent density perturbations at rational-q surfaces following pellet injection in the Joint European Torus". In: *Phys. Rev. Lett.* 59 (20 1987), pp. 2303–2306. DOI: `10.1103/PhysRevLett.59.2303`.

[55] A. L. Pecquet et al. "Snake-like phenomena in Tore Supra following pellet injection". In: *Nucl. Fusion* 37.4 (1997), p. 451. DOI: `https://doi.org/10.1088/0029-5515/37/4/I03`.

[56] W. A. Cooper et al. "Tokamak Magnetohydrodynamic Equilibrium States with Axisymmetric Boundary and a 3D Helical Core". In: *Phys. Rev. Lett.* 105 (3 2010), p. 035003. DOI: `10.1103/PhysRevLett.105.035003`.

[57] W. A. Cooper, J. P. Graves, and O. Sauter. "JET snake magnetohydrodynamic equilibria". In: *Nucl. Fusion* 51.7 (2011), p. 072002. DOI: `https://doi.org/10.1088/0029-5515/51/7/072002`.

[58] R. Lorenzini et al. "Self-organized helical equilibria as a new paradigm for ohmically heated fusion plasmas". In: *Nature Phys.* 5 (2009), pp. 570–574. DOI: `https://doi.org/10.1038/nphys1308`.

[59] W. F. Bergerson et al. "Bifurcation to 3D Helical Magnetic Equilibrium in an Axisymmetric Toroidal Device". In: *Phys. Rev. Lett.* 107 (25 2011), p. 255001. DOI: `10.1103/PhysRevLett.107.255001`.

[60] M. E. Puiatti et al. "Helical equilibria and magnetic structures in the reversed field pinch and analogies to the tokamak and stellarator". In: *Plasma Phys. Control. Fusion* 51.12 (2009), p. 124031. DOI: `https://doi.org/10.1088/0741-3335/51/12/124031`.

[61] D. Terranova et al. "A 3D approach to equilibrium, stability and transport studies in RFX-mod improved regimes". In: *Plasma Phys. Control. Fusion* 52.12 (2010), p. 124023. DOI: `https://doi.org/10.1088/0741-3335/52/12/124023`.

[62] F. L. Waelbroeck. "Theory and observations of magnetic islands". In: *Nucl. Fusion* 49.10 (2009), p. 104025. DOI: `https://doi.org/10.1088/0029-5515/49/10/104025`.

[63] Kōji Uo. "The Confinement of Plasma by the Heliotron Magnetic Field". In: *J. Phys. Soc. Jpn.* 16.7 (1961), pp. 1380–1395. DOI: 10.1143/JPSJ.16.1380.

[64] L. Spitzer. "The Stellarator Concept". In: *Phys. Fluids* 1.4 (1958), pp. 253–264. DOI: 10.1063/1.1705883.

[65] P. Helander et al. "Stellarator and tokamak plasmas: a comparison". In: *Plasma Phys. Control. Fusion* 54.12 (2012), p. 124009. DOI: https://doi.org/10.1088/0741-3335/54/12/124009.

[66] Elisabete M. de Gouveia Dal Pino. "Astrophysical jets and outflows". In: *Adv. Space Res.* 35.5 (2005), pp. 908–924. DOI: https://doi.org/10.1016/j.asr.2005.03.145.

[67] R. E. Pudritz, M. J. Hardcastle, and D. C. Gabuzda. "Magnetic Fields in Astrophysical Jets: From Launch to Termination". In: *Space Sci. Rev.* 169.1–4 (2012), pp. 27–72. DOI: https://doi.org/10.1007/s11214-012-9895-z.

[68] D. A. Kaltsas, G. N. Throumoulopoulos, and P. J. Morrison. "Helically symmetric extended magnetohydrodynamics: Hamiltonian formulation and equilibrium variational principles". In: *Journal of Plasma Physics* 84.3 (2018), p. 745840301. DOI: 10.1017/S0022377818000338.

[69] T. Andreussi, P. J. Morrison, and F. Pegoraro. "MHD equilibrium variational principles with symmetry". In: *Plasma Phys. Control. Fusion* 52.5 (2010), p. 055001. DOI: https://doi.org/10.1088/0741-3335/52/5/055001.

[70] T. Andreussi, P. J. Morrison, and F. Pegoraro. "Hamiltonian magnetohydrodynamics: Helically symmetric formulation, Casimir invariants, and equilibrium variational principles". In: *Phys. Plasmas* 19.5 (2012), p. 052102. DOI: 10.1063/1.4714761.

[71] D. A. Kaltsas, G. N. Throumoulopoulos, and P. J. Morrison. "Translationally symmetric extended MHD via Hamiltonian reduction: Energy-Casimir equilibria". In: *Phys. Plasmas* 24.9 (2017), p. 092504. DOI: 10.1063/1.4986013.

[72] R. D. Hazeltine, C. T. Hsu, and P. J. Morrison. "Hamiltonian four-field model for nonlinear tokamak dynamics". In: *Phys. Fluids* 30.10 (1987), pp. 3204–3211. DOI: 10.1063/1.866527.

[73] Z. Yoshida and E. Hameiri. "Canonical Hamiltonian mechanics of Hall magnetohydrodynamics and its limit to ideal magnetohydrodynamics". In: *J. Phys. A: Math. Theor.* 46.33 (2013), p. 335502. DOI: https://doi.org/10.1088/1751-8113/46/33/335502.

[74] V. I. Ilgisonis. "Equilibrium of flowing plasma in tokamak in the frame of Hall magnetohydrodynamics". In: *Plasma Physics and Controlled Fusion* 43.9 (2001), pp. 1255–1271. DOI: 10.1088/0741-3335/43/9/307.

[75] L. C. Steinhauer and A. Ishida. "Nearby-fluids equilibria. I. Formalism and transition to single-fluid magnetohydrodynamics". In: *Physics of Plasmas* 13.5 (2006), p. 052513. DOI: 10.1063/1.2200610.

[76] E. Hameiri. "Ertel's vorticity theorem and new flux surfaces in multi-fluid plasmas". In: *Physics of Plasmas* 20.9 (2013), p. 092503. DOI: 10.1063/1.4820769.

[77] D. A. Kaltsas, G. N. Throumoulopoulos, and P. J. Morrison. "Ellipticity conditions for the extended MHD Grad-Shafranov-Bernoulli equilibrium equations". In: *Physics of Plasmas* 26.2 (2019), p. 024501. DOI: 10.1063/1.5080997.

[78] S. M. Mahajan and Z. Yoshida. "Double Curl Beltrami Flow: Diamagnetic Structures". In: *Phys. Rev. Lett.* 81 (22 1998), pp. 4863–4866. DOI: 10.1103/PhysRevLett.81.4863.

[79] J. L. Johnson et al. "Some Stable Hydromagnetic Equilibria". In: *Phys. Fluids* 1.4 (1958), pp. 281–296. DOI: 10.1063/1.1705886.

[80] K. C. Tsinganos. "Magnetohydrodynamic equilibrium. III - Helically symmetric fields". In: *Astrophys. J.* 259 (1982), pp. 820–831. DOI: 10.1086/160217.

[81] O. I. Bogoyavlenskij. "Exact Helically Symmetric Plasma Equilibria". In: *Lett. Math. Phys.* 51.4 (2000), pp. 235–247. DOI: 10.1023/A:1007620802075.

[82] G. N. Throumoulopoulos and H. Tasso. "Ideal magnetohydrodynamic equilibria with helical symmetry and incompressible flows". In: *Journal of Plasma Physics* 62.4 (1999), 449–459. DOI: https://doi.org/10.1017/S0022377899008041.

[83] A. Evangelias, A. Kuiroukidis, and G. N. Throumoulopoulos. "Helically symmetric equilibria with pressure anisotropy and incompressible plasma flow". In: *Plasma Phys. Control. Fusion* 60.2 (2018), p. 025005. DOI: https://doi.org/10.1088/1361-6587/aa93a7.

[84] E. Tassi et al. "Hamiltonian formulation and analysis of a collisionless fluid reconnection model". In: *Plasma Phys. Control. Fusion* 50.8 (2008), p. 085014. DOI: https://doi.org/10.1088/0741-3335/50/8/085014.

[85] J. A. Almaguer et al. "Lyapunov stability analysis of magnetohydrodynamic plasma equilibria with axisymmetric toroidal flow". In: *Phys. Fluids* 31.7 (1988), pp. 1930–1939. DOI: 10.1063/1.866640.

[86] T. Andreussi and F. Pegoraro. "On the variational approach to axisymmetric magnetohydrodynamic equilibria". In: *Phys. Plasmas* 15.9 (2008), p. 092108. DOI: 10.1063/1.2980426.

[87] S. M. Moawad. "Linear and nonlinear stability criteria for compressible MHD flows in a gravitational field". In: *J. Plasma Phys.* 79.5 (2013), pp. 873–883. DOI: 10.1017/S0022377813000627.

[88] G. N. Throumoulopoulos and H. Tasso. “On Hall magnetohydrodynamics equilibria”. In: *Phys. Plasmas* 13.10 (2006), p. 102504. DOI: 10.1063/1.2358111.

[89] E. Hameiri. “Ertel’s vorticity theorem and new flux surfaces in multi-fluid plasmas”. In: *Phys. Plasmas* 20.9 (2013), p. 092503. DOI: 10.1063/1.4820769.

[90] L. Guazzotto and R. Betti. “Two-fluid equilibrium with flow: FLOW2”. In: *Phys. Plasmas* 22.9 (2015), p. 092503. DOI: 10.1063/1.4929854.

[91] A. Ishida, L. C. Steinhauer, and Y.-K. M. Peng. “Two-fluid low-collisionality equilibrium model and application to spherical torus plasmas”. In: *Physics of Plasmas* 17.12 (2010), p. 122507. DOI: 10.1063/1.3526600.

[92] A. Ishida and L. C. Steinhauer. “Spherical torus equilibria reconstructed by a two-fluid, low-collisionality model”. In: *Physics of Plasmas* 19.10 (2012), p. 102512. DOI: 10.1063/1.4762846.

[93] W. H. Press et al. *Numerical Recipes in FORTRAN 77: Volume 1, The Art of Scientific Computing*. Numerical Recipes in FORTRAN: The Art of Scientific Computing τ. 2. Cambridge University Press, 1992. ISBN: 9780521430647.

[94] A. Kuiroukidis and G. N. Throumoulopoulos. “Comment on the paper ‘An analytic functional form for characterization and generation of axisymmetric plasma boundaries’”. In: *Plasma Physics and Controlled Fusion* 57 (2015), p. 078001. DOI: 10.1088/0741-3335/57/7/078001.

[95] P. Barberio-Corsetti. “Force-free helical equilibria”. In: *Plasma Phys.* 15.11 (1973), p. 1131. DOI: https://doi.org/10.1088/0032-1028/15/11/007.

[96] S. Chandrasekhar and P. C. Kendall. “On Force-Free Magnetic Fields”. In: *Astrophys. J.* 126 (1957), p. 457. DOI: 10.1086/146413.

[97] Z. Yoshida and S. M. Mahajan. “Variational Principles and Self-Organization in Two-Fluid Plasmas”. In: *Phys. Rev. Lett.* 88 (9 2002), p. 095001. DOI: 10.1103/PhysRevLett.88.095001.

[98] M. Iqbal et al. “High β relaxed states with internal conductor plasma configuration”. In: *Phys. Plasmas* 8.5 (2001), pp. 1559–1564. DOI: 10.1063/1.1364672.

[99] Z. Yoshida et al. “Beltrami fields in plasmas: High-confinement mode boundary layers and high beta equilibria”. In: *Phys. Plasmas* 8.5 (2001), pp. 2125–2131. DOI: 10.1063/1.1354149.

[100] L. C. Evans. *Partial Differential Equations: Second Edition*. Graduate studies in mathematics. American Mathematical Society, 2010. ISBN: 9780821849743.

[101] R. Betti and J. P. Freidberg. “Radial discontinuities in tokamak magnetohydrodynamic equilibria with poloidal flow”. In: *Physics of Plasmas* 7.6 (2000), pp. 2439–2448. DOI: 10.1063/1.874083.

[102] E. Hameiri. “The equilibrium and stability of rotating plasmas”. In: *The Physics of Fluids* 26.1 (1983), pp. 230–237. DOI: 10.1063/1.864012.

[103] J. P. Goedbloed. “Variational principles for stationary one- and two-fluid equilibria of axisymmetric laboratory and astrophysical plasmas”. In: *Physics of Plasmas* 11.12 (2004), pp. L81–L84. DOI: 10.1063/1.1808453.

[104] L. Guazzotto and E. Hameiri. “A model for transonic plasma flow”. In: *Physics of Plasmas* 21.2 (2014), p. 022512. DOI: 10.1063/1.4866600.

[105] A. Ishida et al. “Ellipticity criterion for flowing two-fluid high β equilibria”. In: *Physics of Plasmas* 11.11 (2004), pp. 5297–5301. DOI: 10.1063/1.1804533.

[106] G. I. Hagstrom and E. Hameiri. “Traveling waves in Hall - magnetohydrodynamics and the ion-acoustic shock structure”. In: *Physics of Plasmas* 21.2 (2014), p. 022109. DOI: 10.1063/1.4862035.

[107] A. Ito, J. J. Ramos, and N. Nakajima. “Ellipticity of axisymmetric equilibria with flow and pressure anisotropy in single-fluid and Hall magnetohydrodynamics”. In: *Physics of Plasmas* 14.6 (2007), p. 062502. DOI: 10.1063/1.2741391.

[108] C. B. Morrey. “Second Order Elliptic Systems of Differential Equations”. In: *Proceedings of the National Academy of Sciences* 39.3 (1953), pp. 201–206. DOI: 10.1073/pnas.39.3.201.

[109] D. A. Kaltsas, G. N. Throumoulopoulos, and P. J. Morrison. “Energy-Casimir, dynamically accessible, and Lagrangian stability of extended magnetohydrodynamic equilibria”. In: *submitted to Physics of Plasmas* (2019). URL: https://arxiv.org/abs/1908.08821.

[110] T. Andreussi, P. J. Morrison, and F. Pegoraro. “Hamiltonian magnetohydrodynamics: Lagrangian, Eulerian, and dynamically accessible stability–Examples with translation symmetry”. In: *Phys. Plasmas* 23.10 (2016), p. 102112. DOI: 10.1063/1.4964900.

[111] P. J. Morrison and D. Pfirsch. “The free energy of Maxwell–Vlasov equilibria”. In: *Physics of Fluids B: Plasma Physics* 2.6 (1990), pp. 1105–1113. DOI: 10.1063/1.859246.

[112] E. Tassi, T. S. Ratiu, and E. Lazzaro. “Hamiltonian structure and stability analysis of a reduced four-field model for plasmas in the presence of a strong guide field”. In: *Journal of Physics: Conference Series* 401 (2012), p. 012023. DOI: 10.1088/1742-6596/401/1/012023.

[113] William A Newcomb. “Hydromagnetic stability of a diffuse linear pinch”. In: *Annals of Physics* 10.2 (1960), pp. 232 –267. DOI: https://doi.org/10.1016/0003-4916(60)90023-3.

[114] Eliezer Hameiri. "Variational principles for equilibrium states with plasma flow". In: *Physics of Plasmas* 5.9 (1998), pp. 3270–3281. DOI: 10.1063/1.872995.

[115] M. B. Isichenko. "Nonlinear Hydrodynamic Stability". In: *Phys. Rev. Lett.* 80 (5 1998), pp. 972–975. DOI: 10.1103/PhysRevLett.80.972.

[116] I. Keramidas Charidakos et al. "Action principles for extended magnetohydrodynamic models". In: *Physics of Plasmas* 21.9 (2014), p. 092118. DOI: 10.1063/1.4896336.

[117] W. A. Newcomb. "Lagrangian and Hamiltonian methods in magnetohydrodynamics". In: *Nuclear Fusion Supp.* 2 (1962), p. 451.

[118] M. Vuilemin. "Lagrangian Formalism for a System Composed of Several Fluids Interacting through Electromagnetic Forces". In: (1964). URL: ftp://ftp.aip.org/epaps/phys_fluids_1958-1988/E-PFLDA-31-3024-12.

[119] D. A. Kaltsas and G. N. Throumoulopoulos. "2D magnetofluid models constructed by a priori imposition of conservation laws". In: *Physics Letters A* 383.10 (2019), pp. 1031 –1036. DOI: https://doi.org/10.1016/j.physleta.2018.12.035.

[120] P. Nevir and R. Blender. "A Nambu representation of incompressible hydrodynamics using helicity and enstrophy". In: *Journal of Physics A: Mathematical and General* 26.22 (1993), pp. L1189–L1193. DOI: 10.1088/0305-4470/26/22/010.

[121] R. Salazar and M. V. Kurgansky. "Nambu brackets in fluid mechanics and magnetohydrodynamics". In: *Journal of Physics A: Mathematical and Theoretical* 43.30 (2010), p. 305501. DOI: 10.1088/1751-8113/43/30/305501.

[122] Y. Nambu. "Generalized Hamiltonian Dynamics". In: *Phys. Rev. D* 7 (8 1973), pp. 2405–2412. DOI: 10.1103/PhysRevD.7.2405.

[123] L. Takhtajan. "On foundation of the generalized Nambu mechanics". In: *Communications in Mathematical Physics* 160.2 (1994), pp. 295–315. DOI: 10.1007/BF02103278.

[124] Iwo Bialynicki-Birula and P. J. Morrison. "Quantum mechanics as a generalization of Nambu dynamics to the Weyl-Wigner formalism". In: *Physics Letters A* 158.9 (1991), pp. 453 –457. DOI: https://doi.org/10.1016/0375-9601(91)90458-K.

[125] R. Salmon. "A general method for conserving quantities related to potential vorticity in numerical models". In: *Nonlinearity* 18.5 (2005), R1–R16. DOI: 10.1088/0951-7715/18/5/r01.

[126] R. Salmon. "A General Method for Conserving Energy and Potential Enstrophy in Shallow-Water Models". In: *Journal of the Atmospheric Sciences* 64.2 (2007), pp. 515–531. DOI: 10.1175/JAS3837.1.

[127] R. Blender and G. Badin. "Hydrodynamic Nambu brackets derived by geometric constraints". In: *Journal of Physics A: Mathematical and Theoretical* 48.10 (2015), p. 105501. DOI: 10.1088/1751-8113/48/10/105501.

[128] R. Blender and G. Badin. "Construction of Hamiltonian and Nambu Forms for the Shallow Water Equations". In: *Fluids* 2.2 (2017). DOI: https://doi.org/10.3390/fluids2020024.

[129] A. Thyagaraja. "Conservative regularization of ideal hydrodynamics and magnetohydrodynamics". In: *Physics of Plasmas* 17.3 (2010), p. 032503. DOI: 10.1063/1.3339913.

[130] P. J. Morrison and R. D. Hazeltine. "Hamiltonian formulation of reduced magnetohydrodynamics". In: *The Physics of Fluids* 27.4 (1984), pp. 886–897. DOI: 10.1063/1.864718.

[131] H. R. Strauss. "Nonlinear, three-dimensional magnetohydrodynamics of noncircular tokamaks". In: *The Physics of Fluids* 19.1 (1976), pp. 134–140. DOI: 10.1063/1.861310.

[132] J. L. Thiffeault and P. J. Morrison. "Classification and Casimir invariants of Lie–Poisson brackets". In: *Physica D: Nonlinear Phenomena* 136.3 (2000), pp. 205 –244. DOI: https://doi.org/10.1016/S0167-2789(99)00155-4.

[133] Y. Kaneko and Z. Yoshida. "Canonical Hamiltonian Model of Reduced MHD and its Comparison with the Two-Dimensional Euler System". In: *Plasma and Fusion Research* 8 (2013), pp. 1401057–1401057. DOI: 10.1585/pfr.8.1401057.

[134] D. Grasso et al. "Hamiltonian magnetic reconnection". In: *Plasma Physics and Controlled Fusion* 41.12 (1999), pp. 1497–1515. DOI: 10.1088/0741-3335/41/12/306.

[135] F. L. Waelbroeck and E. Tassi. "A compressible Hamiltonian electromagnetic gyrofluid model". In: *Communications in Nonlinear Science and Numerical Simulation* 17.5 (2012), pp. 2171 –2178. DOI: https://doi.org/10.1016/j.cnsns.2011.04.015.

[136] C. Tronci, E. Tassi, and P. J Morrison. "Energy-Casimir stability of hybrid Vlasov-MHD models". In: *Journal of Physics A: Mathematical and Theoretical* 48.18 (2015), p. 185501. DOI: 10.1088/1751-8113/48/18/185501.

[137] M. Furukawa et al. "Calculation of large-aspect-ratio tokamak and toroidally-averaged stellarator equilibria of high-beta reduced magnetohydrodynamics via simulated annealing". In: *Physics of Plasmas* 25.8 (2018), p. 082506. DOI: 10.1063/1.5038043.

[138] C. Bressan et al. "Relaxation to magnetohydrodynamics equilibria via collision brackets". In: *Journal of Physics: Conference Series* 1125 (2018), p. 012002. DOI: 10.1088/1742-6596/1125/1/012002.

[139] G. N. Throumoulopoulos and H. Tasso. "A sufficient condition for the linear stability of magnetohydrodynamic equilibria with field aligned incompressible flows". In: *Physics of Plasmas* 14.12 (2007), p. 122104. DOI: 10.1063/1.2817957.

[140] M. Kraus, E. Tassi, and D. Grasso. "Variational integrators for reduced magnetohydrodynamics". In: *Journal of Computational Physics* 321 (2016), pp. 435–458. DOI: https://doi.org/10.1016/j.jcp.2016.05.047.